Adventure Is Out There

A Mathematical Exploration of Cartography
Part I

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Summary

How does one chart the Earth?

The Difficult Problem of Cartography

Goal to make a perfect chart of the Earth (which is represented symbolically by M)

Question what is a perfect chart?

Question^2 what is a chart?

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Question: what is a perfect chart?

Question^2: what is a chart?

A Chart

- A piece of plane paper representing (partially or completely) the surface of the Earth?
- The chart is not only the paper: it also depends on the function mapping points on the Earth to points on the paper

In mathematical terms, a chart is a pair (U, φ) where $U \subseteq M$ and

$$\varphi \colon U \to \mathbb{R}^2.$$

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A Perfect Chart

- Each point on the Earth must be mapped to one, and only one, point on the chart;
- each point on the chart must represent one, and only one, point on the Earth;
- points which are close on the Earth must be close on the chart and vice-versa.

A perfect chart is a pair (U, φ) where $U = M$ and $\varphi \colon U \to \mathbb{R}^2$ is a homeomorphism.

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Which space is the Earth?

Taken to be roughly spherical

$$\therefore M = S^2 = \left\{x \in \mathbb{R}^3; \|x\| = 1\right\},$$

where $\|\cdot\|$ is the Euclidean norm

Heine-Borel: A Cartographer's Nightmare

Theorem (Heine-Borel)

Consider the metric space $(\mathbb{R}^n, d)$, where d is the standard Euclidean metric. A subset $K \subseteq \mathbb{R}^n$ is compact if, and only if, it is closed and bounded. □

S^2 is both closed and bounded, so it is compact. $\mathbb{R}^2$ is not bounded, so it is not compact. Therefore, there can't be a homeomorphism between S^2 and $\mathbb{R}^2$

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Heine-Borel: A Cartographer's Nightmare

Do we need to give up on charting the Earth or can we do something about it?

Which axioms can we loose?

Continuity not interesting if all we know is Topology

Bijection not convenient, albeit possible

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- **Continuity** not interesting if all we know is Topology
- **Bijection** not convenient, albeit possible
- **Global** changing to a local description might be interesting, if we are able to get a complete description at the end

Charting the Earth

Current Objective

Make perfect charts of pieces of the Earth that can be “sewn” together continuously and cover the entire Earth

Charting the Earth

Pieces to be considered

We'll chart six pieces of the Earth:

$$U_i^{\pm} \equiv \left\{(x_1, x_2, x_3) \in S^2; \text{sign}(x_i) = \pm 1\right\}.$$

Coordinate maps to be considered

We'll consider the maps $\varphi_1^{\pm} : U_i^{\pm} \to \mathcal{B}_1(0)$

$$\varphi_1^{\pm}\left((x_1, x_2, x_3)\right) = (x_2, x_3),$$

with similar definitions for $\varphi_i^{\pm} : U_i^{\pm} \to \mathcal{B}_1(0)$.

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Charting the Earth

Exercise

Show that the maps $\varphi_i^{\pm}$ are homeomorphisms and that the sets $U_i^{\pm}$ cover S^2.

Or just take a look at [2].

Charting the Earth

Our current description is now made not only through a single chart, but through an atlas.

Issue can we flip the atlas' pages continuously?

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Flipping Pages

If (U, φ) and (V, ψ) are charts with $p \in U \cap V$,
$(\varphi \circ \psi^{-1}) : \psi(U \cap V) \to \varphi(U \cap V)$ is continuous

Crisis on Infinite Earths

Which possible Earths could we chart in this fashion?

Topological Manifolds

Formalizing and Generalizing

Definition (Locally Euclidean Space)

Let (M, τ) be a topological space. We say it is a locally Euclidean space of dimension n if, and only if, every point $p \in M$ has an open neighborhood U which has an homeomorphism φ onto an open subset of $\mathbb{R}^n$. The pair (U, φ) is said to be a chart, U is said to be a coordinate neighborhood and φ is said to be a coordinate system on U. If $\varphi(p) = 0$, the chart (U, φ) is said to be centered at p. ♠

Formalizing and Generalizing

Theorem

Let (M, τ) *be a topological space.* (M, τ) *is a locally Euclidean space if, and only if, every point* $p \in M$ *has an open neighborhood* U *which has an homeomorphism* φ *onto an open ball of* $\mathbb{R}^n$. □

Formalizing and Generalizing

Proof

Suppose every point $p \in M$ has an open neighborhood U which has an homeomorphism φ onto an open ball of $\mathbb{R}^n$. Since every open ball is an open set, it follows immediately that (M, τ) is locally Euclidean.

Formalizing and Generalizing

Proof

Suppose (M, τ) is locally Euclidean. Let $p \in M$. We know there is a pair (U, φ) such that U is an open neighborhood of p and $\varphi\colon U \to \varphi(U)$ is a homeomorphism. Since φ is a homeomorphism, $\varphi(U)$ is an open set. In particular, it is an open neighborhood of $\varphi(p)$. Therefore, there is some $\epsilon > 0$ such that $\varphi(p) \subseteq \mathcal{B}_\epsilon(\varphi(p)) \subseteq \varphi(U)$. Furthermore, since φ is a homeomorphism, $\varphi^{-1}(\mathcal{B}_\epsilon(\varphi(p)))$ is an open neighborhood of p. Also, the restriction of φ to $\varphi^{-1}(\mathcal{B}_\epsilon(\varphi(p)))$ is a homeomorphism, proving that $p \in M$ has an open neighborhood which has an homeomorphism onto an open ball of $\mathbb{R}^n$. ■

Formalizing and Generalizing

Definition (Atlas)

Let (M, τ) be a locally Euclidean space of dimension n. An atlas on (M, τ) is a collection $\mathcal{A} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ of charts on (M, τ) such that $M = \bigcup_{\lambda \in \Lambda} U_\lambda$. ♠

Formalizing and Generalizing

Definition (Topological Manifold)

A topological manifold of dimension n is a Hausdorff, second-countable, locally Euclidean space of dimension n. ♠

Is the dimension of a manifold well-defined?

We assume the following result (which is Corollary 1.6.3 of [14]) without proof:

Theorem (Topological Invariance of Dimension)

Let $n, m \in \mathbb{N}, n > m$. *Let* $\varnothing \neq U \subseteq \mathbb{R}^n$. *There is no continuous injective mapping from* U *to* $\mathbb{R}^m$. *In particular,* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *are not homeomorphic.* □

Is the dimension of a manifold well-defined?

Lemma

Let $n \in \mathbb{N}$, $p \in \mathbb{R}^n$, $\epsilon > 0$. $\mathcal{B}_\epsilon(p)$ *is homeomorphic to* $\mathbb{R}^n$. □

Is the dimension of a manifold well-defined?

Theorem

The dimension of a topological manifold is well-defined. □

Is the dimension of a manifold well-defined?

Proof

Let (M, τ) be a topological manifold of dimension n and assume, for the sake of contradiction, that it is also a topological manifold of dimension $m \neq n$. We assume without any loss of generality that $n > m$.

Is the dimension of a manifold well-defined?

Proof

Let $p \in M$. We know there is an open set U with $p \in U$ and a homeomorphism $\varphi \colon U \to \mathcal{B}_\epsilon(x) \subseteq \mathbb{R}^n$, for some $\epsilon > 0$ and some $x \in \mathbb{R}^n$. Similarly, there is an open set V with $p \in V$ and a homeomorphism $\psi \colon V \to \psi(V) \subseteq \mathcal{B}_\delta(y)$, for some $\delta > 0$ and some $y \in \mathbb{R}^m$. Since open balls in $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}^n$, we know that V is homeomorphic to $\mathbb{R}^m$ (let's call this homeomorphism g) and U is homeomorphic to $\mathbb{R}^n$ (let's call this homeomorphism f).

Is the dimension of a manifold well-defined?

Proof

We may consider the open set $U \cap V$. We know that $f\colon U \cap V \to f(U \cap V) \subseteq \mathbb{R}^n$ is a homeomorphism and so is $g\colon U \cap V \to g(U \cap V) \subseteq \mathbb{R}^m$. Hence, $(g \circ f^{-1})\colon f(U \cap V) \to \mathbb{R}^m$ is a continuous injective map. The Theorem of Topological Invariance of Dimension tells us this is a contradiction. Hence, (M, τ) can't have two different dimensions, proving the dimension of M is well-defined. ■

Is the dimension of a manifold well-defined?

Notation

We denote the dimension of a topological manifold (M, τ) by $\dim M$. ♦

Why these axioms?

- Locally Euclidean if we want to chart the space, it surely need to admit charts
- Hausdorff uniqueness of limits
- Second-Countability provides important extra structure

Why these axioms?

Properties of Topological Manifolds

Let (M, τ) be a topological manifold. It has the following properties[4]

- Locally compact for it is locally Euclidean;
- Separable for it is second-countable;
- Normal for it is Hausdorff, second-countable and locally compact;
- Paracompact for it is Hausdorff, second-countable and locally compact;
- Metrizable for it is normal and second-countable;
- Admits partitions of unity for it is metrizable and separable, and thus, given an atlas $\mathcal{A}$, there is a partition of unity subordinated to the coordinate neighborhoods of $\mathcal{A}$.

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Continuity of Curves

Definition (Continuous Curve)

Let (M, τ) be a topological manifold. A curve on M is a function $\gamma \colon I \to M$, where $I \subseteq \mathbb{R}$. A curve is said to be continuous if it is continuous as a function between topological spaces, where $\mathbb{R}$ is considered to be equipped with the standard topology. Continuity of γ at a point $\lambda \in \mathbb{R}$ is defined in a similar manner. ♠

Continuity of Curves

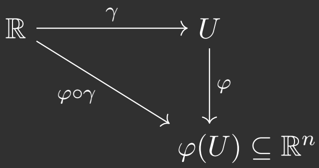
Proposition

Let (M, τ) *be a topological manifold of dimension* n*. Consider a curve* $\gamma \colon \mathbb{R} \to M$*. Let* $\lambda \in \mathbb{R}$ *and consider a chart* (U, φ) *of* M *such that* $\gamma(\lambda) \in U$*.* γ *is continuous at* λ *if, and only if,* $\varphi \circ \gamma$ *is continuous at* λ*.* □

Continuity of Curves

Proof

Suppose γ is continuous at λ. Since φ is a homeomorphism, $\varphi \circ \gamma$ is a composition of continuous functions at x and hence it is continuous. On the other hand, if $\varphi \circ \gamma$ is continuous at x, notice that $\gamma = \varphi^{-1} \circ (\varphi \circ \gamma)$, and hence γ is the composition of continuous functions. ∎



Continuity of Curves

Continuity is chart independent!

Let (U, φ) and (V, ψ) be charts with $\gamma(\lambda) \in U \cap V$. Then $\psi \circ \gamma$ is continuous at λ if, and only if, $\varphi \circ \gamma$ is continuous at λ:

$$\begin{aligned}\psi \circ \gamma &= \psi \circ (\varphi^{-1} \circ \varphi) \circ \gamma, \\ &= (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma).\end{aligned}$$

Since $\psi \circ \varphi^{-1}$ is continuous, continuity of $\varphi \circ \gamma$ implies continuity of $\psi \circ \gamma$.

Continuity of Curves

Definition (Chart Transition Maps)

Let (M, τ) be a topological manifold of dimension n and let (U, φ) and (V, ψ) be charts on M such that $U \cap V \neq \varnothing$. The chart transition maps, or simply transition maps or transition functions, between (U, φ) and (V, ψ) are the maps

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V), \quad \psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V). \ \spadesuit$$

Continuity of Curves

Lemma

Let (M, τ) *be a topological manifold of dimension* n *and let* (U, φ) *and* (V, ψ) *be charts on* M *such that* $U \cap V \neq \varnothing$. *The chart transition maps between* (U, φ) *and* (V, ψ) *are continuous.* □

$\mathcal{C}^k$-Manifolds

Differentiability of Curves

Question How can we define differentiability of a curve?

Issue Topology only deals with continuity, not differentiability

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Differentiability of Curves

Possible solution Define differentiability of $\gamma\colon \mathbb{R} \to M$ based on differentiability of $\varphi \circ \gamma : \mathbb{R} \to \varphi(U) \subseteq \mathbb{R}^n$, for some chart (U, φ)

Issue What if we had another chart (V, ψ)?

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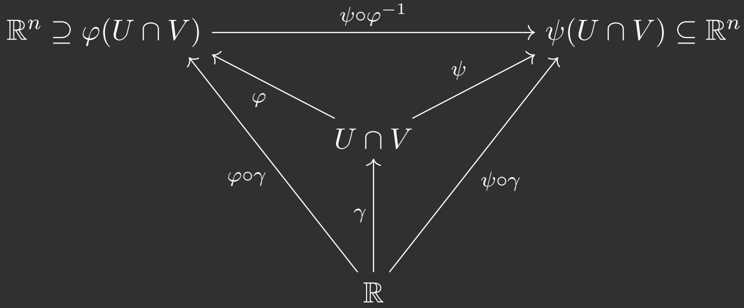
Differentiability of Curves

The definition of differentiability should be chart-independent.



Differentiability of Curves

Definition ($\mathcal{C}^k$-compatible Charts)

Let (M, τ) be a locally Euclidean space. Let (U, φ) and (V, ψ) be charts on (M, τ). The charts are said to be $\mathcal{C}^k$-compatible if, and only if, either of the following requirements hold:

1. $U \cap V = \varnothing$;
2. $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are of class $\mathcal{C}^k$. ♠

Differentiability of Curves

Definition ($\mathcal{C}^k$-atlas)

Let (M, τ) be a locally Euclidean space and let $\mathcal{A}$ be an atlas on (M, τ). $\mathcal{A}$ is said to be a $\mathcal{C}^k$-atlas if, and only if, the charts on $\mathcal{A}$ are pairwise $\mathcal{C}^k$-compatible.
In particular, $\mathcal{C}^\infty$-atlases are commonly referred to as smooth atlases. ♠

Differentiability of Curves

Remark

In order to define differentiability, we are not adding structure. In fact, we are removing. ♣

Differentiability of Curves

Definition (Maximal $\mathcal{C}^k$-atlas)

Let (M, τ) be a topological manifold and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on (M, τ). $\mathcal{A}$ is said to be maximal if, and only if, for every $\mathcal{C}^k$-atlas $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}$ it holds that $\mathcal{A} = \mathcal{A}'$.
A maximal $\mathcal{C}^k$-atlas on a topological manifold (M, τ) is also referred to as a $\mathcal{C}^k$-structure on (M, τ). Once again, the $k = \infty$ case is referred commonly as "smooth" instead of $\mathcal{C}^\infty$. ♠

$\mathcal{C}^k$-manifolds

Definition ($\mathcal{C}^k$-manifold)

Let (M, τ) be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on (M, τ). The triple $(M, \tau, \mathcal{A})$ is said to be a $\mathcal{C}^{[k]}$-manifold.
In particular, $\mathcal{C}^\infty$-manifolds are commonly referred to as smooth manifolds. ♠

Remark

Every $\mathcal{C}^{[k]}$-manifold is a $\mathcal{C}^{[l]}$-manifold for $l \leq k$. ♣

$\mathcal{C}^k$-manifolds

Lemma

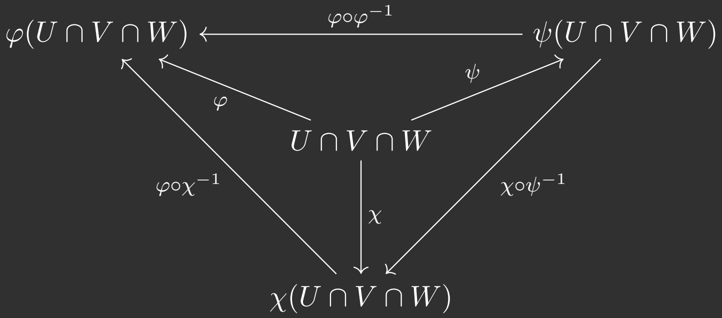
Let (M, τ) be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on (M, τ). Let (U, φ) and (V, ψ) be charts on (M, τ). If both (U, φ) and (V, ψ) are compatible with the atlas $\mathcal{A}$, then they are compatible with each other. □

$\mathcal{C}^k$-manifolds

Proof

If $U \cap V = \varnothing$, the proof is complete. Let us then assume $U \cap V \neq \varnothing$. $\mathcal{A}$ covers M, and therefore, given $p \in U \cap V$, there is some chart (W, χ) with $p \in W$. By hypothesis, (W, χ) is compatible with both (U, φ) and (V, ψ).

$\mathcal{C}^k$-manifolds



$\mathcal{C}^k$-manifolds

Proof

Since (W, χ) is compatible with both (U, φ) and (V, ψ), we know that $\chi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V \cap W)$ and $\varphi \circ \chi^{-1}$ is $\mathcal{C}^k$ at $\chi(U \cap V \cap W)$. Hence, $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V \cap W)$ and, in particular, at $\psi(p)$. Since $p \in U \cap V$ was arbitrary, we see that $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V)$. A similar argument proves that $\psi \circ \varphi^{-1}$ is $\mathcal{C}^k$ at $\varphi(U \cap V)$. Therefore, (U, φ) and (V, ψ) are $\mathcal{C}^k$-compatible. ■

$\mathcal{C}^k$-manifolds

Proposition

Let (M, τ) be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on (M, τ). $\mathcal{A}$ is contained on a unique maximal $\mathcal{C}^k$-atlas. $\square$

$\mathcal{C}^k$-manifolds

Proof

Consider the set $\bar{\mathcal{A}}$ of all charts $\mathcal{C}^k$-compatible with $\mathcal{A}$. Notice that $\mathcal{A} \subseteq \bar{\mathcal{A}}$ and, as a consequence, $\bar{\mathcal{A}}$ is an atlas, for it is a collection of charts that covers M. We must now prove that it is a $\mathcal{C}^k$-atlas and that it is maximal.

$\mathcal{C}^k$-manifolds

Proof

Let $(U, \varphi), (V, \psi) \in \bar{\mathcal{A}}$. By hypothesis, both of them are $\mathcal{C}^k$-compatible with $\mathcal{A}$ and thus are compatible with each other. Therefore, $\bar{\mathcal{A}}$ is a $\mathcal{C}^k$-atlas.

$\mathcal{C}^k$-manifolds

Proof

Suppose now $\mathcal{A}'$ is a $\mathcal{C}^k$-atlas containing $\bar{\mathcal{A}}$. Notice $\mathcal{A} \subseteq \bar{\mathcal{A}} \subseteq \mathcal{A}'$. Thus, every chart (U, φ) in $\mathcal{A}'$ is $\mathcal{C}^k$-compatible with $\mathcal{A}$. Thus, by definition of $\bar{\mathcal{A}}$, every chart (U, φ) of $\mathcal{A}'$ is in $\bar{\mathcal{A}}$, *i.e.*, $\mathcal{A}' \subseteq \bar{\mathcal{A}}$. Therefore, $\bar{\mathcal{A}} = \mathcal{A}'$, proving $\bar{\mathcal{A}}$ is maximal.

$\mathcal{C}^k$-manifolds

Proof

Finally, we must prove $\bar{\mathcal{A}}$ is unique. Suppose $\mathcal{A}'$ is some $\mathcal{C}^k$-atlas with $\mathcal{A} \subseteq \mathcal{A}'$. Then every chart in $\mathcal{A}'$ is compatible with $\mathcal{A}$ and hence $\mathcal{A}' \subseteq \bar{\mathcal{A}}$, so either $\mathcal{A}' = \bar{\mathcal{A}}$ or $\mathcal{A}'$ is not maximal. One way or the other, the proof is complete. ■

$\mathcal{C}^k$-manifolds

When proving some topological space is a $\mathcal{C}^k$-manifold, we do not need to bother with describing the whole maximal atlas. Instead, it suffices to find some atlas and the existence of a maximal atlas is guaranteed.

$\mathcal{C}^k$-manifolds

Remark

A result due to Hassler Whitney states that, for every $k > 0$, a maximal $\mathcal{C}^k$-atlas contains a smooth atlas[7]. As a consequence, one is mostly interested on the theory of smooth manifolds. ♣

Remark

The restriction $k \neq 0$ is important: there are examples of topological manifolds that do not admit a smooth structure. The first example[8] of such a manifold is a compact 10-dimensional manifold constructed by Michel Kervaire in 1960[6]. ♣

$\mathcal{C}^k$-manifolds

Example (Euclidean Space)

The first example of smooth manifold one might consider is $\mathbb{R}^n$ itself, which is a Hausdorff, second-countable space. An atlas is given by $\{(\mathbb{R}^n, \mathrm{id})\}$, where $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ is the function that maps $x \mapsto x$. ♥

$\mathcal{C}^k$-manifolds

Example (Locally Euclidean space which is not Hausdorff)

A simple example of a locally euclidean space which is not Hausdorff is the line with two origins: the real line with an extra point.
We begin by picking some set which we already know to exist. As any set will do, let ❧ denote a leaf. We write $X = \mathbb{R} \cup \{❧\}$.

$\mathcal{C}^k$-manifolds

Example (Locally Euclidean space which is not Hausdorff)

We now proceed to define a topology in X. Let $\mathfrak{B}_{\mathbb{R}}$ be the basis of open intervals for the standard topology in $\mathbb{R}$. Let $\mathfrak{B}_{\text{☙}} \equiv \{\{\text{☙}\} \cup B \setminus \{0\}\,; B \in \mathfrak{B}_{\mathbb{R}}\}$. We define $\mathfrak{B} \equiv \mathfrak{B}_{\mathbb{R}} \cup \mathfrak{B}_{\text{☙}}$. $\mathfrak{B}$ is a basis for a non-Hausdorff topology in X. On the other hand, every point p has an open neighbourhood which can be mapped with the identity (or with a quasi-identity $x \mapsto x$ for $x \neq \text{☙}$ and $\text{☙} \mapsto 0$) to $\mathbb{R}$. Thus, it is locally Euclidean. ♥

$\mathcal{C}^k$-manifolds

Example (2-sphere)

The construction made on the beginning of this chapter can be used to prove that S^2 is a smooth manifold. ♥

Differentiable Maps

Definition ($\mathcal{C}^k$ Maps)

Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds with $\dim M = m$ and $\dim N = n$ and let $p \in M$. A map $f\colon M \to N$ is said to be *of class $\mathcal{C}^k$ at p* if, and only if, there are charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$ and $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$.
The map f is said to be *of class $\mathcal{C}^k$* if, and only if, it is of class $\mathcal{C}^k$ at p for every $p \in M$. A $\mathcal{C}^\infty$ map is often called a *smooth map*. ♠

Differentiable Maps

$$\begin{array}{ccc} M \supseteq U & \xrightarrow{\quad f \quad} & V \subseteq N \\ \varphi \big\downarrow & & \big\downarrow \psi \\ \mathbb{R}^m \supseteq \varphi(U) & \xrightarrow[\psi \circ f \circ \varphi^{-1}]{} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Differentiable Maps

Proposition

The notion of a $\mathcal{C}^k$ map between two $\mathcal{C}^k$-manifolds is well-defined, i.e., it does not depend on the charts chosen. □

Differentiable Maps

Proof

Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds with $\dim M = m$ and $\dim N = n$ and let $p \in M$. Let $f \colon M \to N$ be a map and let there be are charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$ and $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$. We want to show that if there are charts $(W, \zeta) \in \mathcal{A}_M$ and $(X, \xi) \in \mathcal{A}_N$ with $p \in W$ and $f(W) \subseteq X$ such that $\xi \circ f \circ \zeta^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\zeta^{-1}(p)$.

Differentiable Maps

Proof

Notice that $p \in U \cap W$ and $f(U \cap W) \subseteq V \cap X$. We are thus invited to consider the diagram

$$\begin{array}{ccc}
\mathbb{R}^m \supseteq \zeta(U \cap W) & \xrightarrow{\xi \circ f \circ \zeta^{-1}} & \xi(V \cap X) \subseteq \mathbb{R}^n \\
\uparrow \zeta & & \uparrow \xi \\
M \supseteq U \cap W & \xrightarrow{f} & V \cap X \subseteq N \\
\downarrow \varphi & & \downarrow \psi \\
\mathbb{R}^m \supseteq \varphi(U \cap W) & \xrightarrow[\psi \circ f \circ \varphi^{-1}]{} & \psi(V \cap X) \subseteq \mathbb{R}^n
\end{array}$$

Differentiable Maps

Proof

The diagram then invites us to notice that

$$\xi \circ f \circ \zeta^{-1} = (\xi \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \zeta^{-1}),$$

which, due to the fact that $\mathcal{A}_M$ and $\mathcal{A}_N$ are $\mathcal{C}^k$-atlases, is a composition of $\mathcal{C}^k$-maps in the sense of Real Analysis. Hence, $\xi \circ f \circ \zeta^{-1}$ is $\mathcal{C}^k$ in $\zeta(U \cap W)$ and, in particular, in $\zeta(p)$, proving the result. ∎

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