As Time Goes By

Seeking a Solution to Maxwell's Equations

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Summary

1. It's still the same old story
   - Where we look at our theory so far
2. A fight for love and glory
   - When it is time for an update
3. A case of do or die
   - Where we take another point of view
4. The world will always welcome lovers
   - Where we find the couple of retarded potentials
5. As time goes by
   - Where we find Jefimenko's Equations
6. References

Helmholtz Theorem

Theorem (Informal)

If the divergence $(\boldsymbol{\nabla} \cdot \mathbf{F})(\mathbf{r})$ *and the curl* $(\boldsymbol{\nabla} \times \mathbf{F})(\mathbf{r})$ *of a vector function* $\mathbf{F}(\mathbf{r})$ *are specified, and if they both go to zero faster than* $1/r^2$ *as* $r \to \infty$*, and if* $\mathbf{F}(\mathbf{r})$ *goes to zero as* $r \to \infty$*, then* $\mathbf{F}$ *is given uniquely by*

$$\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W},$$

where U *and* $\mathbf{W}$ *are given by*

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \cdot \mathbf{F})(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau' , \quad \mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \times \mathbf{F})(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau' .$$

Not Helmholtz Theorem

Not-a-Theorem

Let $\mathbf{E}$ *be a curl-less field and let* $\mathbf{B}$ *be a divergenceless field. Then we may write them as*

$$\mathbf{E} = -\boldsymbol{\nabla} V, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

where V *and* $\mathbf{A}$ *are given by*

$$V(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \cdot \mathbf{E})(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau' \,, \quad \mathbf{A}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \times \mathbf{B})(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau' \,.$$

Static potentials

Curl-less electric field

$$\nabla \times \mathbf{E} = \mathbf{0}$$

Divergenceless magnetic field

$$\nabla \cdot \mathbf{B} = 0$$

Static potential

$$\mathbf{E} = -\nabla V$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau'$$

Vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t)}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\tau'$$

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Maxwell's Equations

In the beginning, God said

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases} \tag{1}$$

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Dynamic potentials

$\mathbf{B}$ keeps being divergenceless

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Faraday and Helmholtz can be friends

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{A})$$

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}$$

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A new PDE

Bringing Gauss' Law into the game

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}$$

You've unlocked a new equation!

$$\nabla^2 V + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

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Another new PDE

Bringing Ampère-Maxwell's Law into the game

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Time for some hard work

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\boldsymbol{\nabla} V + \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \boldsymbol{\nabla} \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

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$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

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$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

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$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \boldsymbol{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

Vector calculus identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

Now more hard work

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \boldsymbol{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \boldsymbol{\nabla}\left(\mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) - \nabla^2 \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J}$$

Another new PDE

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$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \boldsymbol{\nabla} \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

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You've unlocked a new equation!

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})+\boldsymbol{\nabla}\left(\mu_0\epsilon_0\frac{\partial V}{\partial t}\right)-\nabla^2\mathbf{A}+\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=\mu_0\mathbf{J}$$

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}+\mu_0\epsilon_0\frac{\partial V}{\partial t}\right)-\left(\nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right)=\mu_0\mathbf{J}$$

$$\left(\nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right)-\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}+\mu_0\epsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\mathbf{J}$$

Electrodynamics written with potentials

In the beginning, God said

$$\begin{cases} \nabla^2 V + \dfrac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\dfrac{\rho}{\epsilon_0} \\ \left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \mu_0\epsilon_0 \dfrac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \end{cases} \tag{2}$$

Thinking outside the box

Galilean Transformations

In Classical Mechanics, we may choose a reference system:

- Spatial translations
- Time translations
- Rotations
- Boosts

What are our freedoms?

Can we choose a "reference system" in Electrodynamics?

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The potentials are not unique

Let $\mathbf{A}' := \mathbf{A} + \mathbf{a}$ and $V' := V + b$. Then

- $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}' \Rightarrow \nabla \times \mathbf{a} = \mathbf{0}$
- $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} \Rightarrow \nabla b + \frac{\partial \mathbf{a}}{\partial t} = 0$
- $\mathbf{a} = \nabla \lambda$
- $\nabla b + \frac{\partial \mathbf{a}}{\partial t} = \nabla \left(b + \frac{\partial \lambda}{\partial t} \right) = \mathbf{0}$
- $b = -\frac{\partial \lambda}{\partial t} + \kappa(t)$
- We may let $\kappa(t)$ be part of λ

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Gauge freedom

There is symmetry in the potentials!

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda \\ V' = V - \dfrac{\partial \lambda}{\partial t} \end{cases} \tag{3}$$

We may choose the value of $\boldsymbol{\nabla} \cdot \mathbf{A}$

$$\boldsymbol{\nabla} \cdot \mathbf{A}' = \boldsymbol{\nabla} \cdot \mathbf{A} + \nabla^2 \lambda$$

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$$\boldsymbol{\nabla} \cdot \mathbf{A}' = \boldsymbol{\nabla} \cdot \mathbf{A} + \nabla^2 \lambda$$

Coulomb gauge

Choosing the gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

In the beginning, God said

$$\begin{cases} \nabla^2 V = -\dfrac{\rho}{\epsilon_0} \\ \left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \mu_0 \epsilon_0 \boldsymbol{\nabla}\left(\dfrac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \end{cases} \tag{4}$$

Lorenz gauge

Choosing the gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

In the beginning, God said

$$\begin{cases} \nabla^2 V - \mu_0 \epsilon_0 \dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \end{cases} \tag{5}$$

Reviewing the potentials

What do we have so far?

- How to calculate the static potentials
- The information travels at a speed $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$
- Imagination

Could we calculate the present potentials with retarded times?

$$t_r = t - \frac{R}{c}, \qquad R \equiv \left\| \mathbf{r} - \mathbf{r}' \right\|$$

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Resolution proposal

Static potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t)}{R}\,\mathrm{d}\tau', \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\,\mathrm{d}\tau'$$

Dynamic potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t-\frac{R}{c})}{R}\,\mathrm{d}\tau', \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t-\frac{R}{c})}{R}\,\mathrm{d}\tau'$$

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Can it solve the wave equation?

Riemann's proof (for the scalar potential)

- Let $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$, where $\mathcal{V}$ is the volume in which we integrate and $\mathbf{r} \in \mathcal{V}_1$
- Let V_1 and V_2 be the "partial potentials", *i.e.*,

$$V_i(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}_i} \frac{\rho(\mathbf{r}', t - \frac{R}{c})}{R} \, \mathrm{d}\tau'$$

- Notice that $V = V_1 + V_2$

Can it solve the wave equation?

Riemann's proof (for the scalar potential)

- Let $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$, where $\mathcal{V}$ is the volume in which we integrate and $\mathbf{r} \in \mathcal{V}_1$
- Let V_1 and V_2 be the "partial potentials", *i.e.*,

$$V_i(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}_i} \frac{\rho(\mathbf{r}', t - \frac{R}{c})}{R} \, \mathrm{d}\tau'$$

- Notice that $V = V_1 + V_2$

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- Let $\mathcal{V}_1$ be very small. Then

$$\rho\left(\mathbf{r}', t - \frac{R}{c}\right) \to \rho(\mathbf{r}', t)$$

$$V_1(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}_1} \frac{\rho(\mathbf{r}', t)}{R} \, \mathrm{d}\tau'$$

- V_1 is the static potential! Therefore,

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}$$

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Laplacian in spherical coordinates

$$\nabla^2 \xi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \xi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \xi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \xi}{\partial \phi^2}$$

Riemann's proof (for the scalar potential)

- $R = \|\mathbf{r} - \mathbf{r}'\|$ is spherically symmetric around a fixed $\mathbf{r}'$
- $\rho\left(\mathbf{r}', t - \frac{R}{c}\right)/R$ must be as well
- The Laplacian gets simplified!

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Riemann's proof (for the scalar potential)

- The Laplacian gets simplified!

$$
\begin{aligned}
\nabla^2\left(\frac{\rho}{R}\right) &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{\rho}{R}\right)\right)\\
&= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R\frac{\partial\rho}{\partial R}-\rho\right)\\
&= \frac{1}{R^2}\left(R\frac{\partial^2\rho}{\partial R^2}+\frac{\partial\rho}{\partial R}-\frac{\partial\rho}{\partial R}\right)\\
&= \frac{1}{R}\frac{\partial^2\rho}{\partial R^2}
\end{aligned}
$$

Can it solve the wave equation?

Riemann's proof (for the scalar potential)

- Back to the potentials, we can see that

$$\nabla^2 V_2 = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2} \nabla^2\left(\frac{\rho\left(\mathbf{r}', t-\frac{R}{c}\right)}{R}\right)\mathrm{d}\tau' = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{R}\frac{\partial^2}{\partial R^2}\rho\left(\mathbf{r}', t-\frac{R}{c}\right)\mathrm{d}\tau'$$

- However, a function of the form $u(t-\frac{R}{c})$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

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Riemann's proof (for the scalar potential)

- Luckily, $\rho\left(\mathbf{r}', t - \frac{R}{c}\right)$ has this form! Therefore,

$$\frac{\partial^2 \rho}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = 0$$

- We see now that

$$\nabla^2 V_2 = \frac{1}{4\pi\epsilon_0 c^2}\int_{\mathcal{V}_2} \frac{1}{R}\frac{\partial^2}{\partial t^2}\rho\left(\mathbf{r}', t - \frac{R}{c}\right) \mathrm{d}\tau'$$

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Riemann's proof (for the scalar potential)

- With some mathematical mambo jambo we get that

$$\begin{aligned}\nabla^2 V_2 &= \frac{1}{4\pi\epsilon_0 c^2}\int_{\mathcal{V}_2}\frac{1}{R}\frac{\partial^2}{\partial t^2}\rho\left(\mathbf{r}', t-\frac{R}{c}\right)\mathrm{d}\tau' \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{\rho\left(\mathbf{r}', t-\frac{R}{c}\right)}{R}\mathrm{d}\tau' \\ &= \frac{1}{c^2}\frac{\partial^2 V_2}{\partial t^2}\end{aligned}$$

Can it solve the wave equation?

Riemann's proof (for the scalar potential)

- Finally, if we let $\mathcal{V}_1 \to 0$, then $\mathcal{V}_2 \to \mathcal{V}$ and it will follow that

$$\nabla^2 V_2 = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

- We already know that

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}$$

- Let's add them up!

$$\nabla^2 V_1 + \nabla^2 V_2 = \nabla^2 V_1 + V_2 = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho}{\epsilon_0}$$

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Riemann's proof (for the scalar potential)

- Through this process, we finally get that, indeed,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

- The proof for $\mathbf{A}$ is analogous
- The argument also holds for $t_a = t + \frac{R}{c}$

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Why care about checking?

Intuition seems to fail for the electromagnetic fields

$$\mathbf{E}(\mathbf{r},t) \neq \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{r}',t-\frac{R}{c}\right)}{R^2}\hat{\mathbf{R}}\,\mathrm{d}\tau'$$

$$\mathbf{B}(\mathbf{r},t) \neq \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}',t-\frac{R}{c}\right)\times\hat{\mathbf{R}}}{R^2}\,\mathrm{d}\tau'$$

What did we do wrong?

"Where physical intuition can't go, we bring math, *i.e.*, we bring math almost everywhere."

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Retracing our steps

What did we do right?

- Express the potentials in the Lorenz gauge
- Find the source terms in the wave equations
- Use the knowledge that information propagates at a speed c
- "Solve" the inhomogeneous three-dimensional wave-equation and pretend we didn't notice
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Electromagnetic waves

Exercise

Use Maxwell's Equations and the following identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

to prove that

$$\begin{cases} \nabla^2 \mathbf{E} - \dfrac{1}{c^2} \dfrac{\partial^2 \mathbf{E}}{\partial t^2} = \dfrac{\boldsymbol{\nabla} \rho}{\epsilon_0} + \mu_0 \dfrac{\partial \mathbf{J}}{\partial t} \\ \nabla^2 \mathbf{B} - \dfrac{1}{c^2} \dfrac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J} \end{cases} \tag{6}$$

Back to intuition

Same equation, same rules

The solutions to the wave equations should be (and are, indeed)

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\int \frac{(\boldsymbol{\nabla}\rho)\left(\mathbf{r}', t-\frac{R}{c}\right)}{R}\,\mathrm{d}\tau' - \frac{\mu_0}{4\pi}\int \frac{1}{R}\frac{\partial}{\partial t}\mathbf{J}\left(\mathbf{r}', t-\frac{R}{c}\right)\mathrm{d}\tau'$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{(\boldsymbol{\nabla}\times\mathbf{J})\left(\mathbf{r}', t-\frac{R}{c}\right)}{R}\,\mathrm{d}\tau' \tag{7}$$

The Road to $\mathbb{E}$-dorado

Rewriting $(\boldsymbol{\nabla}\rho)(\mathbf{r}', t_r)$

$$\begin{aligned}\boldsymbol{\nabla}'\rho(\mathbf{r}', t_r) &= (\boldsymbol{\nabla}\rho)(\mathbf{r}', t_r) + \frac{\partial\rho}{\partial t}\boldsymbol{\nabla}' t_r \\ &= (\boldsymbol{\nabla}\rho)(\mathbf{r}', t_r) + \frac{\partial\rho}{\partial t}\boldsymbol{\nabla}'\left(t - \frac{R}{c}\right) \\ &= (\boldsymbol{\nabla}\rho)(\mathbf{r}', t_r) + \frac{1}{c}\frac{\partial\rho}{\partial t}\hat{\mathbf{R}}\end{aligned}$$

The Road to $\mathbf{E}$-dorado

Notation

From now on, we will write $\lfloor \chi \rfloor$ to denote $\chi(\mathbf{r}', t_r)$, *i.e.*, $\lfloor \chi \rfloor$ denotes the retarded χ

Substituting in $\mathbf{E}$

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= -\frac{1}{4\pi\epsilon_0}\int \frac{\lfloor \boldsymbol{\nabla}\rho \rfloor}{R}\,\mathrm{d}\tau' - \frac{\mu_0}{4\pi}\int \frac{\left\lfloor \dot{\mathbf{j}} \right\rfloor}{R}\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int \left[-\frac{\lfloor \boldsymbol{\nabla}'\rho \rfloor}{R} + \frac{\lfloor \dot{\rho} \rfloor}{cR}\hat{\mathbf{R}} - \frac{\left\lfloor \dot{\mathbf{j}} \right\rfloor}{c^2 R} \right] \mathrm{d}\tau'\end{aligned}$$

The Road to $\mathbf{E}$-dorado

Stokes' Theorem

$$\int_{\mathcal{V}} \boldsymbol{\nabla} T \, \mathrm{d}\tau = \oint_{\partial\mathcal{V}} T \, \mathrm{d}\mathbf{S}$$

Back to the gradient

$$\begin{aligned}\int_{\mathcal{V}} \frac{\lfloor \boldsymbol{\nabla}' \rho \rfloor}{R} \, \mathrm{d}\tau' &= \int_{\mathcal{V}} \left[\boldsymbol{\nabla}' \left(\frac{\lfloor \rho \rfloor}{R} \right) - \lfloor \rho \rfloor \, \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \right] \mathrm{d}\tau' \\ &= \oint_{\partial\mathcal{V}} \frac{\lfloor \rho \rfloor}{R} \, \mathrm{d}\mathbf{S}' - \int_{\mathcal{V}} \lfloor \rho \rfloor \, \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \mathrm{d}\tau'\end{aligned}$$

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The Road to $\mathbb{E}$-dorado

Back to the gradient

$$\int_{\mathcal{V}} \frac{\lfloor \boldsymbol{\nabla}' \rho \rfloor}{R} \,\mathrm{d}\tau' = \oint_{\partial\mathcal{V}} \frac{\lfloor \rho \rfloor}{R} \,\mathrm{d}\mathbf{S}' - \int_{\mathcal{V}} \lfloor \rho \rfloor \,\boldsymbol{\nabla}' \left(\frac{1}{R}\right) \mathrm{d}\tau'$$

Since the integration is carried over all space and the charges vanish when $r \to \infty$, it follows that

$$\oint_{\partial\mathcal{V}} \frac{\lfloor \rho \rfloor}{R} \,\mathrm{d}\mathbf{S}' = \mathbf{0}$$

$$\int \frac{\lfloor \boldsymbol{\nabla}' \rho \rfloor}{R} \,\mathrm{d}\tau' = -\int \lfloor \rho \rfloor \,\boldsymbol{\nabla}' \left(\frac{1}{R}\right) \mathrm{d}\tau'$$

$$\int \frac{\lfloor \boldsymbol{\nabla}' \rho \rfloor}{R} \,\mathrm{d}\tau' = -\int \lfloor \rho \rfloor \,\frac{\hat{\mathbf{R}}}{R^2} \,\mathrm{d}\tau'$$

The Road to $\mathbb{E}$-dorado

Jefimenko's Equation for the electric field

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\lfloor\rho\rfloor}{R^2}\hat{\mathbf{R}} + \frac{\lfloor\dot{\rho}\rfloor}{cR}\hat{\mathbf{R}} - \frac{\left\lfloor\dot{\mathbf{J}}\right\rfloor}{c^2R}\right]\mathrm{d}\tau'$$

Who can it B now?

Known solution

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{(\boldsymbol{\nabla}\times\mathbf{J})\left(\mathbf{r}',t-\frac{R}{c}\right)}{R}\,\mathrm{d}\tau'$$

Rewriting $(\boldsymbol{\nabla}\times\mathbf{J})\left(\mathbf{r}',t-\frac{R}{c}\right)$

$$\begin{aligned}
\boldsymbol{\nabla}'\times\left\lfloor\mathbf{J}\right\rfloor &= \left\lfloor\boldsymbol{\nabla}\times\mathbf{J}\right\rfloor + \boldsymbol{\nabla}' t_r \times \left\lfloor\frac{\partial\mathbf{J}}{\partial t_r}\right\rfloor \\
&= \left\lfloor\boldsymbol{\nabla}\times\mathbf{J}\right\rfloor + \frac{1}{c}\hat{\mathbf{R}}\times\left\lfloor\frac{\partial\mathbf{J}}{\partial t_r}\right\rfloor \\
&= \left\lfloor\boldsymbol{\nabla}\times\mathbf{J}\right\rfloor - \frac{\left\lfloor\dot{\mathbf{J}}\right\rfloor}{c}\times\hat{\mathbf{R}}
\end{aligned}$$

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Rewriting $(\nabla \times \mathbf{J})\left(\mathbf{r}', t - \frac{R}{c}\right)$

$$\begin{aligned}\nabla' \times \lfloor \mathbf{J} \rfloor &= \lfloor \nabla \times \mathbf{J} \rfloor + \nabla' t_r \times \left\lfloor \frac{\partial \mathbf{J}}{\partial t_r} \right\rfloor \\ &= \lfloor \nabla \times \mathbf{J} \rfloor + \frac{1}{c}\hat{\mathbf{R}} \times \left\lfloor \frac{\partial \mathbf{J}}{\partial t_r} \right\rfloor \\ &= \lfloor \nabla \times \mathbf{J} \rfloor - \frac{\left\lfloor \dot{\mathbf{J}} \right\rfloor}{c} \times \hat{\mathbf{R}}\end{aligned}$$

Who can it $\mathbf{B}$ now?

Rewriting $(\nabla \times \mathbf{J})\left(\mathbf{r}', t - \frac{R}{c}\right)$

$$\nabla' \times \lfloor \mathbf{J} \rfloor = \lfloor \nabla \times \mathbf{J} \rfloor - \frac{\left\lfloor \dot{\mathbf{J}} \right\rfloor}{c} \times \hat{\mathbf{R}}$$

Substituting in $\mathbf{B}$

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\lfloor \nabla \times \mathbf{J} \rfloor}{R} \, \mathrm{d}\tau' \\ &= \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \lfloor \mathbf{J} \rfloor}{R} + \frac{\left\lfloor \dot{\mathbf{J}} \right\rfloor \times \hat{\mathbf{R}}}{cR} \, \mathrm{d}\tau'\end{aligned}$$

Who can it $\mathbf{B}$ now?

Stokes' Theorem

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \times \mathbf{T} \, \mathrm{d}\tau = - \oint_{\partial \mathcal{V}} \mathbf{T} \times \mathrm{d}\mathbf{S}$$

Back to the curl

$$\begin{aligned} \int_{\mathcal{V}} \frac{\boldsymbol{\nabla}' \times \lfloor \mathbf{J} \rfloor}{R} \, \mathrm{d}\tau' &= \int_{\mathcal{V}} \boldsymbol{\nabla}' \times \left(\frac{\lfloor \mathbf{J} \rfloor}{R} \right) \mathrm{d}\tau' + \int_{\mathcal{V}} \lfloor \mathbf{J} \rfloor \times \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \mathrm{d}\tau' \\ &= - \oint_{\partial \mathcal{V}} \frac{\lfloor \mathbf{J} \rfloor}{R} \times \mathrm{d}\mathbf{S}' + \int_{\mathcal{V}} \lfloor \mathbf{J} \rfloor \times \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \mathrm{d}\tau' \end{aligned}$$

Who can it B now?

Back to the curl

$$\int_{\mathcal{V}} \frac{\boldsymbol{\nabla}' \times \lfloor \mathbf{J} \rfloor}{R}\, \mathrm{d}\tau' = -\oint_{\partial\mathcal{V}} \frac{\lfloor \mathbf{J} \rfloor}{R} \times \mathrm{d}\mathbf{S}' + \int_{\mathcal{V}} \lfloor \mathbf{J} \rfloor \times \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \mathrm{d}\tau'$$

Since the integration is carried over all space and the currents vanish when $r \to \infty$, it follows that

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$$\int \frac{\boldsymbol{\nabla}' \times \lfloor \mathbf{J} \rfloor}{R}\, \mathrm{d}\tau' = \int \lfloor \mathbf{J} \rfloor \times \frac{\hat{\mathbf{R}}}{R^2}\, \mathrm{d}\tau'$$

Who can it B now?

Jefimenko's Equation for the magnetic field

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\lfloor \mathbf{J} \rfloor}{R^2} + \frac{\left\lfloor \dot{\mathbf{J}} \right\rfloor}{cR} \right] \times \hat{\mathbf{R}} \, \mathrm{d}\tau'$$

A solution to Maxwell's Equations

Jefimenko's Equations

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \left[\dfrac{\lfloor\rho\rfloor}{R^2}\hat{\mathbf{R}} + \dfrac{\lfloor\dot{\rho}\rfloor}{cR}\hat{\mathbf{R}} - \dfrac{\left\lfloor\dot{\mathbf{J}}\right\rfloor}{c^2R}\right]\mathrm{d}\tau' \\ \mathbf{B}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \left[\dfrac{\lfloor\mathbf{J}\rfloor}{R^2} + \dfrac{\left\lfloor\dot{\mathbf{J}}\right\rfloor}{cR}\right]\times\hat{\mathbf{R}}\,\mathrm{d}\tau' \end{cases} \tag{8}$$

Starting over

What about the potential formulation?

One might use the retarded potentials and the identities

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

to obtain Jefimenko's Equations

Final remarks

Jefimenko's Equations

- The electromagnetic fields are generated directly by the charges and currents
- The treatment must be more carefull than the one used to obtain the retarded potentials
- Reduce to Coulomb's and Biot-Savart's Laws in the static limit

Physical intuition

- Valuable, but dangerous
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