Vector Calculus from a Geometrical Perspective

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ABSTRACT: This document is a brief exercise on Differential Geometry aimed at proving the usual expressions of vector calculus in curvilinear coordinates by employing Differential Geometry techniques. The definitions and notations are those of Wald 1984.

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Contents

1 Preliminaries

Before we begin with the calculations, we should make some preliminary definitions in order to have a clear understanding of how to compute the gradient, divergence, curl, and Laplacian of scalar and vector fields. While our main focus will be devoted to three-dimensional Euclidean space, this section will be written in such a manner that we can get generic definitions for a pseudo-Riemannian manifold $(\mathcal{M}, g_{ab})$ of dimension n. We'll let s denote the number of negative entries on the metric signature of the manifold.

1.1 Wedge Product

Our first step will be to recall a few definition about differential forms. As discussed on Wald 1984, App. B, a differential k-form is a completely antisymmetric $(0, k)$-type tensor. We'll often omit the "differential" and simply call them k-forms. We define the wedge product between a p-form ω and a k-form μ as the $(p+k)$ form $\omega \wedge \mu$ defined by

$$(\omega \wedge \mu)_{a_1 \cdots a_p b_1 \cdots b_k} = \frac{(p+k)!}{p!k!} \omega_{[a_1 \cdots a_p} \mu_{b_1 \cdots b_k]}. \tag{1.1}$$

1.2 Exterior Derivative

If ω is a k-form, we may define its exterior derivative $\mathrm{d}\omega$ as the $(k+1)$-form

$$(\mathrm{d}\omega)_{a_1 \cdots a_{k+1}} = (k+1) \nabla_{[a_1} \omega_{a_2 \cdots a_{k+1}]}, \tag{1.2}$$

where ∇_a denotes any (torsionless) differential operator on the manifold. The expression is well-defined because the antisymmetrization cancels out the Christoffel symbols, since the torsionless condition implies these are symmetric on the lower indices. As a consequence, we can simply choose $\nabla_a = \partial_a$ for some arbitrary choice of coordinates. For the remainder of this text, ∇_a will always denote the Levi-Civita connection induced by the metric.

Using the antissymetric properties of forms, it can be shown that $\mathrm{d}(\mathrm{d}\omega) = 0$ for any form ω, *id est*, $\mathrm{d}^2 = 0$.

To gain some intuition, let us consider a few examples. Let first f be a zero-form, *id est*, $f \in \mathcal{C}^\infty(\mathcal{M})$. Then

$$(\mathrm{d}f)_a = \partial_a f, \tag{1.3a}$$
$$(\mathrm{d}f)_a n^a = n^a \partial_a f, \tag{1.3b}$$
$$\mathrm{d}f\,(n) = n(f). \tag{1.3c}$$

Notice that $(\mathrm{d}f)_a = \partial_a f$ is quite similar to how we express the gradient in vector calculus. We'll get back to it later.

For the next examples, well specialize to three dimensions, which is where we want to get eventually. Let us then pick F to be a one-form field, *id est*, $F \in \Gamma(T^*\mathcal{M})$. In this case, we'll have, in Cartesian coordinates,

$$\mathrm{d}F = \partial_i F_j \,\mathrm{d}x^i \wedge \mathrm{d}x^j\,, \tag{1.4a}$$
$$= \left(\partial_y F_z - \partial_z F_y\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\partial_z F_x - \partial_x F_z\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\partial_x F_y - \partial_y F_x\right) \mathrm{d}x \wedge \mathrm{d}y\,, \tag{1.4b}$$

whose components resemble the curl of a vector field.

Finally, pick F to be a two-form, *id est*, $F \in \Gamma\left(\bigwedge^2 T^*\mathcal{M}\right)$. Then

$$\mathrm{d}F = \partial_i F_{jk} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\,, \tag{1.5a}$$
$$= \partial_i F_{jk} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\,, \tag{1.5b}$$
$$= \left(\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \tag{1.5c}$$

which resembles the expression for the divergence of a vector field, if we were to write $F^i \propto \epsilon^{ijk} F_{jk}\,.$

1.3 Hodge Dual

We'll also define the Hodge dual of a k-form ω, denoted $\star\omega$, according to

$$\star\,\omega_{b_1 \cdots b_{n-k}} = \frac{1}{k!}\omega^{a_1 \cdots a_k} \epsilon_{a_1 \cdots a_k b_1 \cdots b_{n-k}}, \tag{1.6}$$

where $\epsilon_{a_1 \cdots a_n}$ is the natural volume element on the manifold. One can show that

$$\star\star\omega = (-1)^{s+k(n-k)}\omega. \tag{1.7}$$

1.4 Gradient

Our first step is to define the gradient in a coordinate-invariant manner. Let $f \in \mathcal{C}^\infty(\mathcal{M})$. The gradient $\boldsymbol{\nabla} f$ of f can be seen as a vector whose inner product with any unit vector $\widehat{\mathbf{n}}$ yields the directional derivative $\mathbf{n} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial n}$. Since we are defining a vector in terms of its action on other vectors, it is simpler if we first consider the one-form associated to it by means of the metric isomorphism and only then define the actual vector field.

We want to consider the one-form that to each unit vector $\hat{n}$ assigns the directional derivative $\mathbf{n} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial n}$. By definition, this is done by the function's differential, $\mathrm{d}f$, defined according to $\mathrm{d}f\,(n) = n(f)$. Hence, the gradient is defined as

$$(\boldsymbol{\nabla} f)^a = g^{ab}(\mathrm{d}f)_b = \nabla^a f. \tag{1.8}$$

In index-free notation, we can simply write this vector field as $\mathrm{d}f^\sharp$, where $\sharp$ denotes "a raised index".

1.5 Divergence

Next we want to obtain the expression for the divergence. Let $F \in \Gamma(T\mathcal{M})$, *id est*, let F be a vector field over the manifold. We've seen on Eq. (1.5) on the preceding page that the expression for $\mathrm{d}F$ when F is a two-form field remembers the expression for the divergence if we were to write $F^i \propto \epsilon^{ijk} F_{jk}$. However, we do not want F to be a two-form field, we want it to be a vector. What can we do?

As we've noticed before, the trick is in using a Levi-Civita symbol, which we can generalize to the volume form on the manifold. We then notice that the expression we had at hands really reminds us of a Hodge dual. Hence, let us start with the vector field F. We can obtain a one-form from it by using the metric, leading us to $F_\flat$ (in abstract index notation, $F_a = g_{ab} F^b$). Using the Hodge dual, we then get $\star F_\flat$. We may now take the differential and obtain $\mathrm{d}(\star F_\flat)$. This is a three-form, so we can use the Hodge dual again and get to a scalar by writing $\star\,\mathrm{d}(\star F_\flat)$. Let us see how this object is written in abstract index notation. We have

$$\star \mathrm{d} \star F_\flat = \frac{1}{6!}[\mathrm{d}(\star F_\flat)]^{abc}\epsilon_{abc}, \tag{1.9a}$$

$$= \frac{1}{6!}[\mathrm{d}(\star F_\flat)]_{def} g^{ad} g^{be} g^{cf} \epsilon_{abc}, \tag{1.9b}$$

$$= \frac{1}{2}\partial_{[d}(\star F_\flat)_{ef]} g^{ad} g^{be} g^{cf} \epsilon_{abc}, \tag{1.9c}$$

$$= \frac{1}{2}\nabla_{[d}(\star F_\flat)_{ef]} g^{ad} g^{be} g^{cf} \epsilon_{abc}, \tag{1.9d}$$

$$= \frac{1}{2}\nabla_{[d}\left(F^g \epsilon_{ef]g}\right) g^{ad} g^{be} g^{cf} \epsilon_{abc}, \tag{1.9e}$$

$$= \frac{1}{2}\nabla_{[d} F^g \epsilon_{ef]g} g^{ad} g^{be} g^{cf} \epsilon_{abc}, \tag{1.9f}$$

$$= \frac{1}{2}\nabla_d F^g \epsilon_{efg} \epsilon^{def}, \tag{1.9g}$$

$$= \frac{1}{2}\nabla_d F^g \epsilon_{efg}\epsilon^{def}, \tag{1.9h}$$

$$= \nabla_d F^g \delta_g{}^d, \tag{1.9i}$$

$$= \nabla_g F^g, \tag{1.9j}$$

which, in locally flat coordinates, is precisely

$$\star \mathrm{d} \star F_\flat = \partial_i F^i, \tag{1.10}$$

id est, it is indeed the divergence.

1.6 Curl

As for the curl, we can follow the lead provided by Eq. (1.4) on page 2. Suppose F is a vector field. Then $F_\flat$ is a one-form field and $\mathrm{d}F_\flat$ is a two-form field whose components resemble those of the curl of F. Let us then take the Hodge dual to get to the one-form field $\star\,\mathrm{d}F_\flat$ and at last consider the associated vector field, $(\star\,\mathrm{d}F_\flat)^\sharp$. How does this expression read in abstract index notation?

Once again, we work in a three-dimensional, Riemannian manifold for simplicity. We get

$$(\star\,\mathrm{d}F_\flat)^\sharp = g^{ab}(\star\,\mathrm{d}F_\flat)_b, \tag{1.11a}$$

$$= \frac{1}{2} g^{ab} (\mathrm{d}F_\flat)^{cd} \epsilon_{cdb}, \tag{1.11b}$$

$$= \frac{1}{2} \epsilon^{acd} (\mathrm{d}F_\flat)_{cd}, \tag{1.11c}$$

$$= \epsilon^{acd} \nabla_{[c} \Big(g_{d]e} F^e \Big), \tag{1.11d}$$

$$= \epsilon^{acd} \nabla_c F^e g_{de}. \tag{1.11e}$$

To proceed, we now pick a system of locally flat coordinates on the manifold, which leads us to

$$(\star\,\mathrm{d}F_\flat)^\sharp = \epsilon^{ijk} \partial_j F^l \delta_{lk} (\partial_i)^a, \tag{1.12a}$$

$$= \big(\partial_y F^z - \partial_z F^y\big)(\partial_x)^a + (\partial_z F^x - \partial_x F^z)(\partial_y)^a + \big(\partial_x F^y - \partial_y F^x\big)(\partial_z)^a, \tag{1.12b}$$

which is precisely the expression we expected for the curl.

1.7 Laplacian

At last, let us also consider the Laplacian of both scalar and of a vector fields.

Firstly let $f \in \mathcal{C}^\infty(\mathcal{M})$. In this case, the Laplacian can be defined as the divergence of the gradient. Hence, using the previous formulae we get that the Laplacian can be written as $\star \mathrm{d} \star \mathrm{d}f$. In abstract index notation it will be written as

$$\star \mathrm{d} \star \mathrm{d}f = \nabla_a \nabla^a f. \tag{1.13}$$

As for the Laplacian of a vector field, it can be defined in vector notation as

$$\nabla^2\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}). \tag{1.14}$$

Hence, using our previous results we get to $[\mathrm{d}\star\mathrm{d}\star F_\flat - \star\mathrm{d}\star\mathrm{d}F_\flat]^\sharp$. In abstract index notation this becomes (for a three-dimensional, Riemannian manifold)

$$\begin{aligned}
[\mathrm{d}\star\mathrm{d}\star F_\flat - \star\mathrm{d}\star\mathrm{d}F_\flat]^\sharp &= \nabla^a\nabla_b F^b - \epsilon^{abc}\nabla_b\Big(\epsilon_c{}^{de}\nabla_d F_e\Big), &(1.15a)\\
&= \nabla^a\nabla_b F^b - \epsilon^{abc}\nabla_b\Big(\epsilon_c{}^{de}\nabla_d F_e\Big), &(1.15b)\\
&= \nabla^a\nabla_b F^b - \epsilon^{abc}\nabla_b\Big(\epsilon_c{}^{de}\nabla_d F_e\Big), &(1.15c)\\
&= \nabla^a\nabla_b F^b - 2\delta^{[a}{}_d\delta^{b]}{}_e\nabla_b\nabla^d F^e, &(1.15d)\\
&= \nabla^a\nabla_b F^b - \nabla_b\nabla^a F^b + \nabla_b\nabla^b F^a, &(1.15e)\\
&= -R^a{}_b F^b + \nabla_b\nabla^b F^a, &(1.15f)
\end{aligned}$$

where the last step uses known properties of the Riemann and Ricci tensors (see Wald 1984, pp. 37–39).

2 Cylindrical Coordinates

Let us compute a few of the common vector calculus identities in cylindrical coordinates. Given the large number of covariant derivatives occurring on the expressions we'll be dealing with, we better begin by writing down the metric and its Christoffel symbols. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2\,. \tag{2.1}$$

One can then compute the Christoffel symbols and find out the non-vanishing ones are given by or related to

$$\Gamma^r{}_{\theta\theta} = -r \quad\text{and}\quad \Gamma^\theta{}_{r\theta} = \frac{1}{r}. \tag{2.2}$$

Before we continue, let us notice that in differential geometry we often work with non-normalized bases, but in vector notation this is not the case. For example, the vector $(\partial_\theta)^a$ is not normalized, since

$$(\partial_\theta)^a(\partial_\theta)^b g_{ab} = r^2. \tag{2.3}$$

However, $\frac{1}{r}(\partial_\theta)^a$ is normalized. Hence, we'll write

$$F^a = F^r(\partial_r)^a + \frac{F^\theta}{r}(\partial_\theta)^a + F^z(\partial_z)^a \tag{2.4}$$

so we are able to compare our formulae with the standard one from, *exempli gratia*, Electrodynamics textbooks, such as Griffiths 2017.

At last, the curl will require us to be able to write the volume form on cylindrical coordinates. We recall that this can be done by usual coordinate transformation methods and yields

$$\epsilon_{abc} = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = r\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}z\,. \tag{2.5}$$

Raising the indices, we get to

$$\epsilon^{abc} = 3! r g^{ad} g^{be} g^{cf} (\mathrm{d}r)_{[d} (\mathrm{d}\theta)_e (\mathrm{d}z)_{f]}, \tag{2.6a}$$

$$= \frac{6}{r} (\partial_r)^{[a} (\partial_\theta)^b (\partial_z)^{c]}. \tag{2.6b}$$

2.1 Gradient

We begin with the gradient. It reads

$$(\boldsymbol{\nabla} f)^a = \nabla^a f, \tag{2.7a}$$

$$= \partial^a f, \tag{2.7b}$$

$$= g^{ab} \partial^b f, \tag{2.7c}$$

$$= [\partial_r f (\mathrm{d}r)_b + \partial_\theta f (\mathrm{d}\theta)_b + \partial_z f (\mathrm{d}z)_b] g^{ab}, \tag{2.7d}$$

$$= \partial_r f (\partial_r)_b + \frac{\partial_\theta f}{r^2} (\partial_\theta)_b + \partial_z f (\partial_z)_b] g^{ab}. \tag{2.7e}$$

Taking into account that we want the components with respect to the normalized basis given by

$$\hat{\mathbf{r}} = (\partial_r)^a, \quad \hat{\boldsymbol{\theta}} = \frac{1}{r} (\partial_\theta)^a, \quad \text{and} \quad \hat{\mathbf{z}} = (\partial_z)^a, \tag{2.8}$$

we find that

$$\boldsymbol{\nabla} f = \partial_r f \hat{\mathbf{r}} + \frac{1}{r} \partial_\theta f \hat{\boldsymbol{\theta}} + \partial_z f \hat{\mathbf{z}}. \tag{2.9}$$

2.2 Divergence

Let us now consider the divergence of a vector field. Using the convention of Eq. (2.4) on the preceding page in which the components are always given in terms of the orthonormal basis we have

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \nabla_a F^a, \tag{2.10a}$$

$$= \partial_a F^a + \Gamma^a{}_{ab} F^b, \tag{2.10b}$$

$$= \partial_r F^r + \partial_\theta \left(\frac{F^\theta}{r} \right) + \partial_z F^z + \Gamma^\theta{}_{\theta r} F^r, \tag{2.10c}$$

$$= \partial_r F^r + \partial_\theta \left(\frac{F^\theta}{r} \right) + \partial_z F^z + \frac{1}{r} F^r, \tag{2.10d}$$

$$= \frac{1}{r} \partial_r (r F^r) + \partial_\theta \left(\frac{F^\theta}{r} \right) + \partial_z F^z, \tag{2.10e}$$

which is precisely the expected result.

2.3 Curl

Next we consider the curl. Since we want to compute $(\boldsymbol{\nabla} \times \mathbf{F})^a = \epsilon^{abc}\nabla_b F_c$, we start by noticing that

$$F_c = F^r(\mathrm{d}r)_c + rF^\theta(\mathrm{d}\theta)_c + F^z(\mathrm{d}z)_c. \tag{2.11}$$

As a consequence,

$$\begin{aligned} \nabla_b F_c &= \partial_b F_c - \Gamma^d{}_{bc} F_d\,, && (2.12\text{a}) \\ &= \partial_b F_c - \Gamma^d{}_{bc} F_d\,, && (2.12\text{b}) \end{aligned}$$

and hence

$$\begin{aligned} \epsilon^{abc}\nabla_b F_c &= \epsilon^{abc}\Big[\partial_b F_c - \Gamma^d{}_{bc} F_d\Big], && (2.13\text{a}) \\ &= \epsilon^{abc}\partial_b F_c\,, && (2.13\text{b}) \end{aligned}$$

due to the torsionless condition. Therefore, we now got to

$$\begin{aligned} (\boldsymbol{\nabla} \times \mathbf{F})^a &= \epsilon^{abc}\partial_b F_c\,, && (2.14\text{a}) \\ &= \frac{6}{r}(\partial_r)^{[a}(\partial_\theta)^b(\partial_z)^{c]}(\partial_b F_c), && (2.14\text{b}) \\ &= \frac{6}{r}(\partial_r)^{[a}(\partial_\theta)^b(\partial_z)^{c]}(\partial_b F_c), && (2.14\text{c}) \\ &= \frac{1}{r}\Big[(\partial_\theta F^z - \partial_z(rF^\theta))(\partial_r)^a + (\partial_z F^r - \partial_r F^z)(\partial_\theta)^a + (\partial_r(rF^\theta) - \partial_\theta F^r)(\partial_z)^a\Big], && (2.14\text{d}) \end{aligned}$$

finally leading us to the conclusion that

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial_\theta F^z}{r} - \partial_z F^\theta\right)\hat{\mathbf{r}} + (\partial_z F^r - \partial_r F^z)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\partial_r(rF^r) - \frac{\partial_\theta F^r}{r}\right)\hat{\mathbf{z}}, \tag{2.15}$$

as expected.

2.4 Scalar Laplacian

Next let us consider the scalar Laplacian. Notice it is different from the vector Laplacian, since the covariant derivatives will act differently on each type of tensor. It is only in Cartesian coordinates that they both coincide. We'll begin by working out the Laplacian for scalar functions. We have

$$\begin{aligned} \nabla^2 f &= \nabla_a \nabla^a f, && (2.16\text{a}) \\ &= g^{ab}\nabla_a \partial_b f, && (2.16\text{b}) \\ &= g^{ab}\partial_a \partial_b f - g^{ab}\Gamma^c{}_{ab}\partial_c f, && (2.16\text{c}) \\ &= \partial_r^2 f + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f - \frac{1}{r^2}\Gamma^c{}_{\theta\theta}\partial_c f, && (2.16\text{d}) \\ &= \partial_r^2 f + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f + \frac{1}{r}\partial_r f, && (2.16\text{e}) \\ &= \frac{1}{r}\partial_r(r\partial_r f) + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f, && (2.16\text{f}) \end{aligned}$$

as expected.

2.5 Vector Laplacian

Finally we consider the vector Laplacian. Since we're working in flat Euclidean space, the Ricci curvature vanishes and we can write

$$\left(\nabla^2 \mathbf{F}\right)^a = \nabla_b \nabla^b F^a, \tag{2.17}$$

meaning the big change in the expression is the fact that now the connection acts in a more complicated way.

References

Griffiths, David J. (2017). *Introduction to Electrodynamics.* Cambridge: Cambridge University Press.

Wald, Robert M. (1984). *General Relativity.* Chicago: University of Chicago Press.