Stress-Energy Tensor for a Scalar Field

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Abstract: This is a computation of the stress-energy-momentum tensor for a scalar field with non-minimal coupling to the background curved spacetime. The goal is mainly to keep the result ready at hand when I need it again in the future, so parts of the computation come from references instead of being redone entirely.

Keywords: General Relativity, Klein–Gordon fields, stress-energy-momentum tensor.

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1 Action and Conventions

We'll consider the field to have action given by

$$S_M = -\frac{1}{2}\int \left(\nabla^a\varphi\nabla_a\varphi + V(\varphi) + \xi R\varphi^2\right)\sqrt{-g}\,\mathrm{d}^4x\,, \tag{1.1}$$

where typically one will be interested in the case $V(\varphi) = m^2\varphi^2$.

Our conventions are essentially those of Wald 1984, but we'll write the integrals in terms of coordinates instead of using differential forms. While this procedure introduces a coordinate system without necessity, the expressions should be more familiar to those more acquainted with field theory. Also, we'll immediately correct Wald 1984, Eq. (E.1.26) to read

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}}, \tag{1.2}$$

which essentially means we're already setting $\alpha_M = 16\pi$ as Wald 1984, p. 455 instructs the reader to do for the case of a Klein–Gordon field, or simply that we're considering the full action (including gravity and the matter fields) to read

$$S_M = \int \left[\frac{1}{16\pi}R - \frac{1}{2}\left(\nabla^a\varphi\nabla_a\varphi + V(\varphi) + \xi R\varphi^2\right)\right]\sqrt{-g}\,\mathrm{d}^4x\,. \tag{1.3}$$

2 Computation

We need to compute*

$$\frac{\delta}{\delta g^{ab}}\left[\int \left(\nabla^a\varphi\nabla_a\varphi + V(\varphi) + \xi R\varphi^2\right)\sqrt{-g}\,\mathrm{d}^4x\right]. \tag{2.1}$$

Notice that we could write the kinetic term in terms of ordinary derivatives if we preferred, so it doesn't really depend on the metric apart from the contraction and the

*Notice that while we took the $-\frac{1}{2}$ factor away from Eq. (2.1), this would need to be done in Eq. (1.2) either way.

volume element. Hence, we have

$$\frac{\delta}{\delta g^{ab}}\left[\int g^{ab}\nabla_a\varphi\nabla_b\varphi\sqrt{-g}\,\mathrm{d}^4x\right] = \nabla_a\varphi\nabla_b\varphi\sqrt{-g} + \nabla^c\varphi\nabla_c\varphi\frac{\delta\sqrt{-g}}{\delta g^{ab}}, \tag{2.2a}$$

$$= \nabla_a\varphi\nabla_b\varphi\sqrt{-g} - \frac{1}{2}g_{ab}\nabla^c\varphi\nabla_c\varphi\sqrt{-g}, \tag{2.2b}$$

where we employed Wald 1984, Eq. (E.1.17),

$$\frac{\delta\sqrt{-g}}{\delta g^{ab}} = -\frac{1}{2}\sqrt{-g}g_{ab}. \tag{2.3}$$

Next we move to the potential. It is given simply by

$$\frac{\delta}{\delta g^{ab}}\left[\int V(\varphi)\sqrt{-g}\,\mathrm{d}^4x\right] = V(\varphi)\frac{\delta\sqrt{-g}}{\delta g^{ab}}, \tag{2.4a}$$

$$= -\frac{1}{2}g_{ab}V(\varphi)\sqrt{-g}. \tag{2.4b}$$

Finally, we deal with the curvature coupling. It is given by

$$\frac{\delta}{\delta g^{ab}}\left[\int \xi R\varphi^2\sqrt{-g}\,\mathrm{d}^4x\right] = \xi\frac{\delta R_{cd}}{\delta g^{ab}}g^{cd}\varphi^2\sqrt{-g} + \xi R_{cd}\frac{\delta g^{cd}}{\delta g^{ab}}\varphi^2\sqrt{-g} + \xi R\varphi^2\frac{\delta\sqrt{-g}}{\delta g^{ab}}, \tag{2.5a}$$

$$= \xi\frac{\delta R_{cd}}{\delta g^{ab}}g^{cd}\varphi^2\sqrt{-g} + \xi R_{ab}\varphi^2\sqrt{-g} - \frac{1}{2}Rg_{ab}\xi\varphi^2\sqrt{-g}, \tag{2.5b}$$

$$= \xi\frac{\delta R_{cd}}{\delta g^{ab}}g^{cd}\varphi^2\sqrt{-g} + \xi G_{ab}\varphi^2\sqrt{-g}. \tag{2.5c}$$

Hence, so far we have the expression

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}[\nabla_c\varphi\nabla^c\varphi + V(\varphi)] + \xi\left(G_{ab} + \frac{\delta R_{cd}}{\delta g^{ab}}g^{cd}\right)\varphi^2. \tag{2.6}$$

Therefore, all that remains is to compute the term associated with the Ricci tensor.

To compute the term associated with the Ricci tensor, it can be easier to do it from inside an integral, so we can explicitly see how we can carry a few integrations by parts. Hence, we begin with

$$\int \xi g^{cd}\delta R_{cd}\varphi^2\sqrt{-g}\,\mathrm{d}^4x\,. \tag{2.7}$$

From Wald 1984, Eq. (3.4.5), we know that

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho{}_{\mu\nu} - \partial_\mu\Gamma^\rho{}_{\rho\nu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{\rho\nu}\Gamma^\rho{}_{\sigma\mu}. \tag{2.8}$$

In a locally inertial reference frame, we'll find that

$$\delta R_{\mu\nu} = \partial_\rho\delta\Gamma^\rho{}_{\mu\nu} - \partial_\mu\delta\Gamma^\rho{}_{\rho\nu}, \tag{2.9a}$$

$$= \nabla_\rho\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\mu\delta\Gamma^\rho{}_{\rho\nu}. \tag{2.9b}$$

Since the last equation is tensorial ($\delta\Gamma^{\rho}{}_{\mu\nu}$ is a difference of Christoffel symbols, and hence a tensor — see Poisson 2004, p. 122), it holds in all coordinate systems, and we conclude that

$$\delta R_{ab} = \nabla_c \delta\Gamma^c{}_{ab} - \nabla_a \delta\Gamma^c{}_{cb}, \tag{2.10a}$$
$$g^{ab}\delta R_{ab} = g^{ab}\nabla_c \delta\Gamma^c{}_{ab} - g^{ab}\nabla_a \delta\Gamma^c{}_{cb}, \tag{2.10b}$$
$$= g^{ab}\nabla_c \delta\Gamma^c{}_{ab} - g^{cb}\nabla_c \delta\Gamma^a{}_{ab}, \tag{2.10c}$$
$$= \nabla_c \Big[g^{ab}\delta\Gamma^c{}_{ab} - g^{cb}\delta\Gamma^a{}_{ab} \Big]. \tag{2.10d}$$

Next we need to compute the variation of the Christoffel symbols. From a similar argument, we have

$$\delta\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}\delta\big(g^{\sigma\tau}\big(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}\big)\big), \tag{2.11a}$$
$$= \frac{1}{2}\delta g^{\sigma\tau}\big(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}\big) + \frac{1}{2}g^{\sigma\tau}\big(\partial_\mu \delta g_{\nu\tau} + \partial_\nu \delta g_{\tau\mu} - \partial_\tau \delta g_{\mu\nu}\big), \tag{2.11b}$$
$$= \frac{1}{2}\delta g^{\sigma\tau}\big(\nabla_\mu g_{\nu\tau} + \nabla_\nu g_{\tau\mu} - \nabla_\tau g_{\mu\nu}\big) + \frac{1}{2}g^{\sigma\tau}\big(\nabla_\mu \delta g_{\nu\tau} + \nabla_\nu \delta g_{\tau\mu} - \nabla_\tau \delta g_{\mu\nu}\big), \tag{2.11c}$$
$$= \frac{1}{2}g^{\sigma\tau}\big(\nabla_\mu \delta g_{\nu\tau} + \nabla_\nu \delta g_{\tau\mu} - \nabla_\tau \delta g_{\mu\nu}\big), \tag{2.11d}$$
$$\delta\Gamma^c{}_{ab} = \frac{1}{2}g^{cd}(\nabla_a \delta g_{bd} + \nabla_b \delta g_{da} - \nabla_d \delta g_{ab}). \tag{2.11e}$$

We'll be interested in two versions of this expression to substitute in the expression for the variation of the Ricci tensor. The first of them is

$$g^{ab}\delta\Gamma^c{}_{ab} = \frac{1}{2}g^{ab}g^{cd}(\nabla_a \delta g_{bd} + \nabla_b \delta g_{da} - \nabla_d \delta g_{ab}), \tag{2.12a}$$
$$= \frac{1}{2}g^{ab}g^{cd}(2\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab}). \tag{2.12b}$$

The remaining one is

$$g^{cb}\delta\Gamma^a{}_{ab} = \frac{1}{2}g^{cb}g^{ad}(\nabla_a \delta g_{bd} + \nabla_b \delta g_{da} - \nabla_d \delta g_{ab}), \tag{2.13a}$$
$$= \frac{1}{2}g^{cb}g^{ad}\nabla_b \delta g_{da}. \tag{2.13b}$$

With these results in mind, we now see that

$$g^{ab}\delta R_{ab} = \nabla_c \Big[g^{ab}\delta\Gamma^c{}_{ab} - g^{cb}\delta\Gamma^a{}_{ab} \Big], \tag{2.14a}$$
$$= \frac{1}{2}\nabla_c \Big[g^{ab}g^{cd}(2\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab}) - g^{cb}g^{ad}\nabla_b \delta g_{da} \Big], \tag{2.14b}$$
$$= g^{ab}g^{cd}\nabla_c\nabla_a \delta g_{bd} - \frac{1}{2}g^{ab}\nabla_c\nabla^c \delta g_{ab} - \frac{1}{2}g^{ad}\nabla_c\nabla^c \delta g_{ad}, \tag{2.14c}$$
$$= g^{ab}g^{cd}\nabla_c\nabla_a \delta g_{bd} - g^{ab}\nabla_c\nabla^c \delta g_{ab}. \tag{2.14d}$$

Recalling now that $\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}$ (see Wald 1984, p. 453), we see that

$$g^{ab}\delta R_{ab} = g_{ab}\nabla_c\nabla^c\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab}, \tag{2.15a}$$
$$= (g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)\delta g^{ab}. \tag{2.15b}$$

We then get to

$$\int \xi g^{ab}\delta R_{ab}\varphi^2\sqrt{-g}\,\mathrm{d}^4x = \int \xi\varphi^2(g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)\delta g^{ab}\sqrt{-g}\,\mathrm{d}^4x\,, \tag{2.16a}$$
$$= \int \xi\delta g^{ab}(g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)(\varphi^2)\sqrt{-g}\,\mathrm{d}^4x + \text{surface terms}. \tag{2.16b}$$

We'll discard the surface terms and conclude that

$$\xi\frac{1}{\sqrt{-g}}\frac{\delta R_{cd}}{\delta g^{ab}}g^{cd}\varphi^2 = \xi(g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)(\varphi^2). \tag{2.17}$$

Finally, we find that we can write the stress-energy-tensor as

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}[\nabla_c\varphi\nabla^c\varphi + V(\varphi)] + \xi\varphi^2G_{ab} + \xi(g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)(\varphi^2). \tag{2.18}$$

3 Trace of the Stress-Energy-Momentum Tensor

As a final check, it is useful to compute the trace of the tensor we just found so we can see, for example, whether we get a traceless tensor in the case $V = 0$, $\xi = \frac{d-2}{4(d-1)}$, which corresponds to a conformal theory (see Wald 1984, App. D). In the previous expression and in the following computations, d corresponds to the dimension of spacetime.

Firstly, let us notice that the action of Eq. (1.1) on page 1 leads us to the equations of motion

$$\nabla_a\nabla^a\varphi - \frac{1}{2}V'(\varphi) - \xi R\varphi = 0, \tag{3.1}$$

where $V'(\varphi)$ is the derivative of V with respect to φ.

Also, let us notice that, in d dimensions, the trace of the Einstein tensor is

$$g^{ab}\left(R_{ab} - \frac{1}{2}Rg_{ab}\right) = \left(1 - \frac{d}{2}\right)R. \tag{3.2}$$

Now let us begin computing the trace. We have

$$T \equiv g^{ab}T_{ab}\,, \tag{3.3a}$$
$$= \nabla_a\varphi\nabla^a\varphi - \frac{d}{2}[\nabla_c\varphi\nabla^c\varphi + V(\varphi)] + \xi\varphi^2G_a{}^a + \xi(d\nabla_c\nabla^c - \nabla_a\nabla^a)(\varphi^2), \tag{3.3b}$$
$$= \left(1 - \frac{d}{2}\right)\nabla_a\varphi\nabla^a\varphi - \frac{d}{2}V(\varphi) + \xi\varphi^2\left(1 - \frac{d}{2}\right)R + \xi(d-1)\nabla_a\nabla^a(\varphi^2). \tag{3.3c}$$

We now notice that

$$\nabla_a \nabla^a (\varphi^2) = 2\nabla_a (\varphi \nabla^a \varphi), \tag{3.4a}$$

$$= 2\nabla_a \varphi \nabla^a + 2\varphi \nabla_a \nabla^a. \tag{3.4b}$$

Hence, the trace becomes

$$T = \left[2\xi(d-1) - \frac{d-2}{2} \right] \nabla_a \varphi \nabla^a \varphi - \frac{d}{2} V(\varphi) - \frac{\xi(d-2)}{2} R\varphi^2 + 2\xi(d-1)\varphi \nabla_a \nabla^a. \tag{3.5}$$

If we now employ the equations of motion and simplify the expressions, we find that

$$T = \left[2\xi(d-1) - \frac{d-2}{2} \right] \left(\nabla_a \varphi \nabla^a \varphi + \xi R \varphi^2 \right) - \frac{d}{2} V(\varphi) + \xi(d-1)\varphi V'(\varphi). \tag{3.6}$$

For $\xi = \frac{d-2}{4(d-1)}$ and $V(\varphi) = 0$, the expression reduces to $T = 0$, just as we expected.

References

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