June 15, 2020

Locally Convex Spaces and the Tempered Distributions

Níckolas Alves

Department of Mathematical Physics,
Institute of Physics - University of São Paulo,
R. do Matão 1371,
Cidade Universitária, São Paulo, Brazil

NUSP: 10300894

E-mail: nickolas@fma.if.usp.br

ABSTRACT: This text is a brief discussion about some elementary topics on the theory of locally convex spaces and tempered distributions as continuous linear functionals on a particular LCS: the Schwartz space of functions of rapid decrease. It is part of the evaluation of the course "Mathematical Physics III", taught by Prof. Walter de S. Pedra at IFUSP on the first semester of 2020. Familiarity with general topology and linear topological spaces is assumed.

KEYWORDS: Distribution Theory, Locally Convex Spaces, Functional Analysis

Contents

1 Tempered Distributions

The goal of this text is to understand tempered distributions under the light of the theory of locally convex spaces. In order to do so, let us begin by defining what we mean by a tempered distribution.

Definition 1 [Functions of Rapid Decrease and Schwartz Space]**:**
Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. f is said to be a *function of rapid decrease* if, and only if, it holds that

$$\sup_{x\in\mathbb{R}^n} |p(x)D^\alpha f(x)| < +\infty \tag{1.1}$$

for every polynomial $p(x) = p(x_1, \ldots, x_n)$ and every multiindex $\alpha \in \mathbb{N}_0^n$.
The collection of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ of rapid decrease is named *Schwartz space* and usually denoted by $\mathscr{S}(\mathbb{R}^n)$ - or simply $\mathscr{S}$. ♠

This definition is often stated in a different way. [1, 2], for example, require

$$\lim_{\|x\|\to+\infty} p(x)D^\alpha f(x) = 0, \tag{1.2}$$

which is equivalent to the condition we required on Eq. (1.1).
Since f is smooth, all of its derivatives are continuous. The image of a compact set through a continuous function is compact, and by the Heine-Borel Theorem this implies the image of any closed, bounded set $K \subseteq \mathbb{R}^n$ under $p(x)D^\alpha f(x)$ is a closed, bounded subset of $\mathbb{R}$. As a consequence, $p(x)D^\alpha f(x)$ can only escape to infinity when $\|x\| \to +\infty$. Non-constant polynomials always escape to infinity for $\|x\| \to +\infty$. Therefore, if $D^\alpha f(x) \to \lambda$ for some non-vanishing λ, then $p(x)D^\alpha f(x)$ will escape to infinity at $\|x\| \to +\infty$. If $p(x)D^\alpha f(x) \to \lambda \neq 0$ for $\|x\| \to +\infty$, then $x \cdot p(x)D^\alpha f(x)$ will escape to infinity. Thus, Eq. (1.2) is implied by Eq. (1.1). If Eq. (1.2) holds, then the fact that $p(x)D^\alpha f(x)$ can only escape to infinity when $\|x\| \to +\infty$ ensures Eq. (1.1).

Proposition 2:
Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. f is of rapid decrease if, and only if,

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right| < +\infty, \forall\, \alpha, \beta \in \mathbb{N}_0^n, \tag{1.3}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $x^\alpha \equiv \prod_{i=1}^n x_i^{\alpha_i}$. □

Proof:

Assume $f \in \mathscr{S}$ and let $\alpha \in \mathbb{N}_0^n$. Then $\sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right| < +\infty$ for every polynomial $p(x) = p(x_1, \dots, x_n)$ and every multiindex $\beta \in \mathbb{N}_0^n$. Pick the polynomial $p(x) = x^\alpha$. It follows that $\|f\|_{\alpha,\beta} < +\infty$.

Let us assume now that $\|f\|_{\alpha,\beta} < +\infty, \forall\, \alpha, \beta \in \mathbb{N}_0^n$. Notice that a polynomial in n variables $p(x)$ can always be written in the form $p(x) = \sum_{i=1}^m a_i x^{\alpha_i}$ for some $m \in \mathbb{N}$, appropriate multiindices $\alpha_i \in \mathbb{N}_0^n$ and coefficients $a_i \in \mathbb{R}$. Notice that given $\beta \in \mathbb{N}_0^n$, it holds $\forall\, x \in \mathbb{R}^n$ that

$$\begin{aligned} \left|p(x)D^\beta f(x)\right| &= \left|\sum_{i=1}^m a_i x^{\alpha_i} D^\beta f(x)\right|, \\ &\leqslant \sum_{i=1}^m |a_i|\left|x^{\alpha_i} D^\beta f(x)\right|. \end{aligned} \tag{1.4}$$

If we take the supremum on each side, it follows that

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right| &\leqslant \sup_{x\in\mathbb{R}^n} \left\{\sum_{i=1}^m |a_i|\left|x^{\alpha_i} D^\beta f(x)\right|\right\}, \\ &\leqslant \sum_{i=1}^m |a_i| \sup_{x\in\mathbb{R}^n} \left|x^{\alpha_i} D^\beta f(x)\right|. \end{aligned} \tag{1.5}$$

Hence, $\sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right|$ is less than or equal to a finite sum of finite terms. Therefore, $\sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right| < +\infty$, and we conclude $f \in \mathscr{S}$. ∎

Theorem 3:

Consider the space $\mathscr{S}$ of functions of rapid decrease. Let $f, g \in \mathscr{S}$, $\lambda \in \mathbb{R}$. The following statements hold

i. $\|\lambda \cdot f\|_{\alpha,\beta} = |\lambda| \cdot \|f\|_{\alpha,\beta}, \forall\, \alpha, \beta \in \mathbb{N}_0^n$;

ii. $\|f + g\|_{\alpha,\beta} \leqslant \|f\|_{\alpha,\beta} + \|g\|_{\alpha,\beta}, \forall\, \alpha, \beta \in \mathbb{N}_0^n$;

iii. *$\mathscr{S}$ is a real vector space;*

iv. $\|f\|_{\alpha,\beta} = 0, \forall\, \alpha, \beta \in \mathbb{N}_0^n \Rightarrow f = 0$. □

Proof:

Let $\lambda \in \mathbb{R}$. Notice that, for any $f \in \mathscr{S}$,

$$\begin{aligned} \|\lambda \cdot f\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta (\lambda \cdot f)(x)\right|, \\ &= \sup_{x\in\mathbb{R}^n} \left|\lambda \cdot x^\alpha D^\beta f(x)\right|, \\ &= \sup_{x\in\mathbb{R}^n} |\lambda|\left|x^\alpha D^\beta f(x)\right|, \\ &= |\lambda| \cdot \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right|, \\ &= |\lambda| \cdot \|f\|_{\alpha,\beta}. \end{aligned} \tag{1.6}$$

Let now $g \in \mathscr{S}$. With f defined as before, we see that

$$\begin{aligned} \|f + g\|_{\alpha,\beta} &= \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta (f + g)(x) \right|, \\ &= \sup_{x \in \mathbb{R}^n} \left| x^\alpha \left(D^\beta f(x) + D^\beta g(x) \right) \right|. \end{aligned} \tag{1.7}$$

Notice that, $\forall\, x \in \mathbb{R}^n$, it holds that

$$\left| x^\alpha D^\beta f(x) + x^\alpha D^\beta g(x) \right| \leqslant \left| x^\alpha D^\beta f(x) \right| + \left| x^\alpha D^\beta g(x) \right|. \tag{1.8}$$

Thus, we may take the supremum on both sides and see that

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) + x^\alpha D^\beta g(x) \right| &\leqslant \sup_{x \in \mathbb{R}^n} \left[\left| x^\alpha D^\beta f(x) \right| + \left| x^\alpha D^\beta g(x) \right| \right], \\ \|f + g\|_{\alpha,\beta} &\leqslant \sup_{x \in \mathbb{R}^n} \left[\left| x^\alpha D^\beta f(x) \right| + \left| x^\alpha D^\beta g(x) \right| \right], \\ &\leqslant \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| + \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta g(x) \right|, \\ &= \|f\|_{\alpha,\beta} + \|g\|_{\alpha,\beta}. \end{aligned} \tag{1.9}$$

This not only proves that $\|\lambda \cdot f\|_{\alpha,\beta} = |\lambda| \cdot \|f\|_{\alpha,\beta}$ and $\|f + g\|_{\alpha,\beta} \leqslant \|f\|_{\alpha,\beta} + \|g\|_{\alpha,\beta}$, but also that $\mathscr{S}$ is closed under pointwise addition and multiplication by scalar, since finiteness of $\|f\|_{\alpha,\beta}$ and $\|g\|_{\alpha,\beta}$ implies finiteness of $\|f + \lambda \cdot g\|_{\alpha,\beta}$. Since $0 \in \mathscr{S}$ - for constant functions are smooth and $\|0\|_{\alpha,\beta} = 0$ - and the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth functions $f \colon \mathbb{R}^n \to \mathbb{R}$ is a real vector space, we see $\mathscr{S}$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$.

Suppose $f \in \mathscr{S}$ is such that $\|f\|_{\alpha,\beta} = 0, \forall\, \alpha, \beta \in \mathbb{N}_0^n$. Then, in particular, it holds that $\|f\|_{0,0} = 0$, where the multiindex 0 should be understood as the n-uple $(0, \ldots, 0)$. This means that $\sup_{x \in \mathbb{R}^n} |f(x)| = 0$. Since $\sup_{x \in \mathbb{R}^n} |f(x)| \geqslant |f(y)| \geqslant 0, \forall\, y \in \mathbb{R}^n$, we conclude $f(y) = 0, \forall\, y \in \mathbb{R}^n$. ■

Definition 4 [Tempered Distribution]**:**

Consider the space $\mathscr{S}$ of functions of rapid decrease and let $\varphi \colon \mathscr{S} \to \mathbb{R}$ be a linear functional. φ is said to be a *tempered distribution* if, and only if, it holds for every function $f \in \mathscr{S}$ that

$$\lim_{n \to +\infty} \varphi(f_n) = \varphi(f) \tag{1.10}$$

for every sequence $(f_n)_{n \in \mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ with the property that

$$\lim_{n \to \infty} \|f_n - f\|_{\alpha,\beta} = 0. \tag{1.11}$$

♠

This definition is equivalent to the one found in [1], but we chose a slightly different collection of seminorms. [1] uses the functions

$$\|f\|_{m,\alpha} = \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^m |D^\alpha f(x)| \tag{1.12}$$

as seminorms, which can make the proof for Proposition 2 a bit more complicated, since $\|x\|$ is not a n-variable polynomial due to the square root in

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{1.13}$$

[3] uses the same choice of seminorms we made, which uses a multiindex instead of a non-negative integer, but uses the polynomial x^α instead of $(1 + \|x\|)^m$.

It is common to denote the action of a tempered distribution φ on a function f through $\langle \varphi, f \rangle$.

2 Formulation in Locally Convex Spaces

Let us now turn our attention to the theory of locally convex spaces. Theorem 3 motivates us to define the concept of a seminorm.

Definition 5 [Seminorm]**:**

Let $(V, \mathbb{F}, +, \cdot)$ be a vector space, with $\mathbb{F}$ being either the real line or the complex plane. A function $\|\cdot\| \colon V \to \mathbb{R}_+$ is said to be *seminorm* on $(V, \mathbb{F}, +, \cdot)$ whenever the following conditions hold, $\forall\, x, y \in V$:

i. $\|x + y\| \leqslant \|x\| + \|y\|$ (triangle inequality);

ii. $\|\alpha \cdot x\| = |\alpha|\|x\|, \forall\, \alpha \in \mathbb{F}$. ♠

One should notice the functions $\|\cdot\|_{\alpha,\beta}$ defined on the Schwartz space are seminorms.

From a topological point of view, the notion of seminorm is closely related to that of a pseudometric. Norms are known to induce metrics. Seminorms are almost norms, but they fail to ensure that $\|x\| = 0 \Rightarrow x = 0$. Thus, the “not-a-metric” induced by them fails to ensure $d(x, y) \Rightarrow x = y$.

Definition 6 [Pseudometric Space]**:**

Let M be a set and $d \colon M \times M \to \mathbb{R}_+$ be a function satisfying the following axioms, $\forall\, x, y, z \in M$,

i. $d(x, x) = 0$;

ii. $d(x, y) = d(y, x)$;

iii. $d(x, y) \leqslant d(x, z) + d(y, z)$.

(M, d) is said to be a *pseudometric space* and d is said to be a *pseudometric* on M. ♠

Proposition 7:

Let V be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\|\cdot\|$ be a seminorm on V. If we define $d \colon V \to \mathbb{R}_+$ by $d(x, y) := \|x - y\|$, (V, d) is a pseudometric space. (V, d) is a metric space if, and only if, $\|\cdot\|$ is a norm, id est, *if it holds that $\|x\| = 0 \Rightarrow x = 0$.* □

An issue arises when dealing with topologies generated by pseudometrics (and hence, with topologies generated by seminorms): the Hausdorff condition is not ensured in general unless we have a metric.

Theorem 8:

Let (M, d) be a pseudometric space. Let, $\forall\, \epsilon > 0, \forall\, x \in M$,

$$\mathcal{B}_\epsilon(x) = \{y \in M; d(x, y) < \epsilon\}. \tag{2.1}$$

The set

$$\mathfrak{B} = \{\mathcal{B}_\epsilon(x); \epsilon > 0, x \in M\} \tag{2.2}$$

is a basis for a topology on M. Furthermore, the topology generated by $\mathfrak{B}$ is Hausdorff if, and only if, (M, d) is a metric space. □

Corollary 9:

Let V be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\|\cdot\|$ be a seminorm on V. Let τ be the topology induced by $\|\cdot\|$. (V, τ) is Hausdorff if, and only if, $\|\cdot\|$ is a norm. □

The Hausdorff axiom is equivalent to the statement that every net defined on the space admits at most one limit point, *id est*, to the uniqueness of limits of nets. Hence, we may state the following result.

Corollary 10:

Let V be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\|\cdot\|$ be a seminorm on V. Let τ be the topology induced by $\|\cdot\|$. $\|\cdot\|$ is a norm if, and only if, every net $\{x_\alpha\}_{\alpha\in I}$ defined on V has at most one limit point. □

The problem we are having with the present construction is the fact that we are not providing information on how to separate points. Since $d(x, y) = 0$ doesn't imply that $x \neq y$, we have no criterion to topologically distinguish two given points.

We can solve this issue without requiring a norm if we equip the space with multiple seminorms (and as a consequence multiple pseudometrics) and impose a condition on the family of seminorms that allows us to distinguish points from a topological point of view.

Definition 11 [Separate Points]**:**

Let V be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let Λ be a family of indices. For every $\lambda \in \Lambda$, let $\|\cdot\|_\lambda$ be a seminorm on V. The family $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ of seminorms is said to *separate points* if, and only if, $\|x\|_\lambda = 0, \forall\, \lambda \in \Lambda \Rightarrow x = 0$. ♠

Fortunately, as stated in Theorem 3, the seminorms $\|\cdot\|_{\alpha,\beta}$ separate points.

With these definitions, we might now define what is a locally convex space.

Definition 12 [Locally Convex Space]**:**

Let X be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let Λ be a family of indices. Let $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ be a family of seminorms on X that separates points. $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ (which we will usually denote simply as X, whenever $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is well understood) is said to be a *locally convex space*. The *natural topology* on such a space (which is always supposed to be equipped in it, unless we state otherwise) is the weak topology that maintains all of the seminorms, the addition of vectors and the multiplication by scalars continuous. ♠

We must notice that this construction does ensure the Hausdorff axiom.

Theorem 13:

Every locally convex space is Hausdorff. □

Proof:

Let $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ be a locally convex space. For each $\lambda \in \Lambda, y \in X$ let $\rho_{\lambda,y}(x) \equiv \|x-y\|_\lambda, \forall\, x \in X$. Every such function $\rho_{y,\lambda}$ is a continuous function. Indeed, $\rho_{y,\lambda}$ is the composition of the functions $\|\cdot\|_\lambda$ and $x \mapsto x - y$, which are both continuous in the natural topology by hypothesis.

Let $x, y \in X, x \neq y$. We know that there is $\lambda \in \Lambda$ such that $\|x-y\|_\lambda \neq 0$ (otherwise, since the family of seminorms separates points, it would hold that $x = y$). Let $\epsilon = \frac{1}{3}\|x-y\|_\lambda$. Consider the sets $O = \rho^{-1}_{x,\lambda}([0,\epsilon))$ and $U = \rho^{-1}_{y,\lambda}([0,\epsilon))$. Since $[0,\epsilon)$ is open in the relative topology on $\mathbb{R}_+$ and the functions $\rho_{z,\lambda}$ are continuous, O and U are open. Since $\rho_{z,\lambda}(z) = \|z-z\|_\lambda = 0, \forall\, z \in X$, it holds that $x \in O$ and $y \in U$. We want to prove that $O \cap U = \varnothing$. Suppose $z \in O \cap U$.

Since $z \in O$, it holds that $\|x-z\|_\lambda < \epsilon$. Since $z \in U$, it also holds that $\|y-z\|_\lambda < \epsilon$. The triangle inequality yields us that

$$\begin{aligned} 3\epsilon &= \|x-y\|_\lambda, \\ &\leqslant \|x-z\|_\lambda + \|y-z\|_\lambda, \\ &< 2\epsilon, \end{aligned} \tag{2.3}$$

which is a contradiction. Therefore, we conclude $\nexists\, z \in O \cap U$, *id est*, $O \cap U = \varnothing$. This proves that X is indeed a Hausdorff space. ■

If we equip $\mathscr{S}$ with the weak topology generated by the seminorms $\|\cdot\|_{\alpha,\beta}$, by vector addition and multiplication by scalar, then $\mathscr{S}$ is, by construction, a locally convex space.

The name "locally convex space" is motivated by the fact that such spaces admit a system of nuclei - *id est*, a neighborhood basis at the origin - comprised exclusively of convex sets[3].

Proposition 14:

Let $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ *be a locally convex space. Let*

$$N_{\lambda_1,\ldots,\lambda_n;\epsilon} \equiv \left\{x \in X; \|x\|_{\lambda_i} < \epsilon, 1 \leqslant i \leqslant n\right\}. \tag{2.4}$$

The set $\mathfrak{N} = \{N_{\lambda_1,\ldots,\lambda_n;\epsilon}; \lambda_i \in \Lambda, 1 \leqslant i \leqslant n, \epsilon > 0\}$ *is a system of nuclei for the natural topology on* X. □

Let us recall an important result from General Topology[4].

Theorem 15:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f\colon X \to Y$ be a function. f is continuous at $x \in X$ if, and only if, the net $(f(x_\alpha))_{\alpha \in I}$ converges to $f(x)$ for every net $(x_\alpha)_{\alpha \in I}$ converging to x. □

With this at our hands, we may state and prove the following result.

Proposition 16:

Let $(X, \{\|\cdot\|_\lambda\}_{\lambda \in \Lambda})$ be a locally convex space. Let $(x_\alpha)_{\alpha \in I}$ be a net. Given a point $x \in X$, it holds that $x_\alpha \to x$ if, and only if, $\|x_\alpha - x\|_\lambda \to 0, \forall \lambda \in \Lambda$. □

Proof:

- $\Rightarrow$: Suppose $x_\alpha \to x$. Theorem 15 guarantees that, since translations are homeomorphisms on any linear topological space, $x_\alpha - x \to 0$. Since $\|\cdot\|_\lambda$ is continuous, $\forall \lambda \in \Lambda$, it holds, also by Theorem 15, that $\|x_\alpha - x\|_\lambda \to 0, \forall \lambda \in \Lambda$.

- $\Leftarrow$: Let $N_{\lambda_1, \ldots, \lambda_n; \epsilon} \equiv \left\{x \in X; \|x\|_{\lambda_i} < \epsilon, \forall i \in \{i\}_{i=1}^n\right\}$ and

$$\mathfrak{N} = \{N_{\lambda_1, \ldots, \lambda_n; \epsilon} \in \tau; \lambda_i \in \Lambda, \forall i \in \{i\}_{i=1}^n, \epsilon > 0\}. \tag{2.5}$$

  Proposition 14 guarantees $\mathfrak{N}$ is a system of nuclei for the natural topology on X. We also know that Theorem 15 guarantees that, since translations are homeomorphisms on any linear topological space, $x_\alpha - x \to 0 \Leftrightarrow x_\alpha \to x$.

  Since $\|x_\alpha - x\|_\lambda \to 0, \forall \lambda \in \Lambda$, we know that, given $\epsilon > 0$ and $\lambda \in \Lambda, \exists \alpha_\lambda \in I; \left\|x_\beta - x\right\|_\lambda \in [0, \epsilon), \forall \beta > \alpha_\lambda$. As a consequence, we see that given $N \in \mathfrak{N}$, $\exists \alpha \in I; x_\beta - x \in N, \forall \beta > \alpha$. This is due to the fact that given $\alpha, \beta \in I, \exists \gamma \in I; \gamma > \alpha, \beta$. Since N is determined by finitely many indices λ_i, we can use induction to find $\alpha \in I$ with $\alpha > \alpha_{\lambda_i}$ for every i.

  Since every neighborhood of 0 can be written in terms of such sets $N \in \mathfrak{N}$, it follows that $x_\alpha - x \to 0$, proving the result. ■

Corollary 17:

Consider a sequence $(f_n)_{n \in \mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$. Given a function $f \in \mathscr{S}$, it holds that $\lim_{n \to +\infty} f_n = f$ in the natural topology of $\mathscr{S}$ if, and only if, $\lim_{n \to +\infty} \|f_n - f\|_{\alpha, \beta} = 0, \forall \alpha, \beta \in \mathbb{N}_0^n$. □

In general, the topology generated by a family of seminorms won't be that of a metric space. Nevertheless, there are some conditions under which this holds[3].

Theorem 18:

Let X be a locally convex space. The following statements are equivalent:

i. *X is metrizable;*

ii. *the topology on X is generated by a countable family of seminorms.* □

The seminorms $\|\cdot\|_{\alpha, \beta}$ generating the topology on the Schwartz space are indexed by elements of the set $\mathbb{N}_0^n \times \mathbb{N}_0^n = \mathbb{N}_0^{2n}$, which is a finite Cartesian product of a countable set. Hence, it is also countable. We see then that the topology on $\mathscr{S}$ is generated by a countable family of seminorms, and as a consequence we conclude $\mathscr{S}$ is metrizable.

Theorem 19:

A linear functional $\varphi\colon \mathscr{S} \to \mathbb{R}$ *is a tempered distribution if, and only if, it is continuous in the natural topology of* $\mathscr{S}$*.* □

Proof:

Let us begin by assuming φ is a tempered distribution. Given a function $f \in \mathscr{S}$, we know

$$\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle \tag{2.6}$$

for every sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ with the property that

$$\lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0. \tag{2.7}$$

Corollary 17 tells us such sequences are precisely the sequences with $f_n \to f$ in the natural topology of $\mathscr{S}$. Therefore, we know $\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle$ for every sequence $(f_n)_{n\in\mathbb{N}}$ converging to f.

We know $\mathscr{S}$ is metrizable, and thus there is a metric $d\colon \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+$ that generates the natural topology on $\mathscr{S}$. The statement on Theorem 15 can be weakend in a metric space[4]: a function $\psi\colon \mathscr{S} \to \mathbb{R}$ is continuous at a point $g \in \mathscr{S}$ if, and only if, the *sequence* $(\langle \psi, g_n \rangle)_{n\in\mathbb{N}}$ converges to $\langle \psi, g \rangle$ for every *sequence* $(g_n)_{n\in\mathbb{N}}$ converging to g.

Since $\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle$ for every sequence $(f_n)_{n\in\mathbb{N}}$ converging to f, we conclude φ is continuous at f in the natural topology of $\mathscr{S}$. Since the argument holds for every $f \in \mathscr{S}$, φ is continuous.

One should notice that all results used in this proof were equivalences, and thus the same argument holds in the opposite direction. Therefore, if $\varphi\colon \mathscr{S} \to \mathbb{R}$ is a continuous linear functional, it is a tempered distribution. ■

Corollary 20:

The topological dual $\mathscr{S}'$ *of the Schwartz space* $\mathscr{S}$ *is precisely the space of tempered distributions.* □

References

1. Barata, J. C. A. *Notas de Física-Matemática* http://denebola.if.usp.br/~jbarata/Notas_de_aula/notas_de_aula.html.
2. Braga, C. L. R. *Notas de Física-Matemática: Equações Diferenciais, Funções de Green e Distribuições* (eds Wrezinski, W. F., Perez, J. F., Marchetti, D. H. U. & Barata, J. C. A.) (Ed. Livraria da Física, São Paulo, 2006).
3. Reed, M. & Simon, B. *Methods of Modern Mathematical Physics Vol. 1: Functional Analysis* (Academic Press, New York, 1972).
4. Simon, B. *A Comprehensive Course in Analysis, Part 1: Real Analysis* (American Mathematical Society, Providence, 2015).