Geometric Methods of Theoretical Physics

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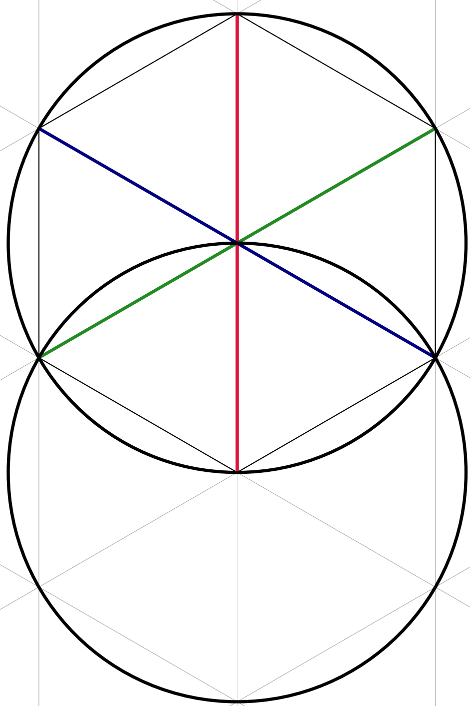

Preface

Years ago I convinced myself that one of the main issues of an undergraduate course in Physics was the major risk that someone could graduate without being exposed to some Differential Geometry. As a relativist, I'm certainly have a few personal reasons to believe that, but I also believe Differential Geometry is a powerful tool in Theoretical Physics even if we pretend to forget about its role in General Relativity. For example, it can give us deep insights in Classical Mechanics, Thermodynamics, Electrodynamics, and certainly in even more cases.

Eventually, I got the courage to start putting together a course in Differential Geometry for physicists. This means I'll try to keep the material as self-contained as possible, but still assuming the readers are familiar with stuff common to Physics, such as Multivariable Calculus and Linear Algebra. Nevertheless, I shall review a few important things as well, since they are excellent ways to build up more complicated stuff, such as tensors. On the other hand, I shall often use examples from man branches of Physics, so some knowledge of Classical Mechanics, Electrodynamics, Quantum Mechanics, and Statistical Mechanics is welcome.

I should mention that, being particularly interested in Mathematics, I might be a bit more rigorous than it is usual for Physics texts. Working with General Relativity strengthened my belief that rigour is often very useful to better understand Physics, even if not used all the time. To use the words of Sommerfeld 1949,

> We do not really deal with mathematical physics, but with physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned "prestabilized harmony" between what is mathematically interesting and what is physically important is met at each step and lends an esthetic I should like to say metaphysical attraction to our subject.

Naturally, since this is not intended to be a text in pure mathematics, I'll allow myself to be sloppy in a few places when being rigorous seems to be pointless. Hopefully, this style should allow both more physical and more mathematical readers to gain from this text and have ease in finding references that fill in the holes I'll leave throughout the way.

Preface

I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments, *et cetera* through my e-mail (alves.nickolas@ufabc.edu.br). If you wish to check some more works, please check my personal website https://alves-nickolas.github.io. In case you are reading this document in a distant future in which I've already concluded my MSc project, you might want to check my ORCID iD (https://orcid.org/0000-0002-0309-735X) for updated contact information. I should mention, though, that this is an ongoing project. Hopefully I will finish it someday.

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One

Why Care About Geometry?

> Let no one ignorant of geometry enter here.
>
> Said to be the sign over the entrance to PLATO's Academy.

I believe any text in Physics or Mathematics should begin by justifying its existence, and hence I guess the natural start for these notes is to answer the question: "why should physicists care about geometry?" A good first answer could be "because coordinate systems are human-made."

In many, if not all, areas of Physics one has often to deal with making good choices of coordinate systems. For example, when solving the equations of motion for a simple pendulum, one could choose to study a constrained system in Cartesian, or a more simple system in polar coordinates. The Physics, of course, is the same, but the approaches may differ and have different advantages. This is made clear in a few formulations of Classical Mechanics. Indeed, one of the great advantages of the Lagrangian formalism is to be able to choose whichever coordinate system one prefers to deal with, while the Hamiltonian approach takes it even further and allows for one to perform canonical transformations in order to make a system more easy to deal with.

The freedom of choice of coordinates suggests that one could reformulate the theory in terms of some structure which "would be there independent of the physicist". While choosing the particular coordinate system to tackle the problem of an oscillating pendulum is up to the physicist, the dynamical properties of the pendulum should not be that arbitrary. To study the dynamics without the need to refer to particular choices of coordinate systems could provide us with deeper insights into what Classical Mechanics is about.

This, of course, is not restrained to Classical Mechanics only. Another example of a physical theory that involves notions of coordinate invariance is General Relativity, which is formulated in geometric terms. As John Wheeler put it, "spacetime tells matter how to move; matter tells spacetime how to curve", and curvature is certainly a geometric notion. The freedom of choosing coordinates can in fact allow one to better understand a few physical consequences of Relativity than can be hard to grasp through other approaches.

For example, that the relativity of time and space is nothing but the fact that different observers have different "natural" coordinate choices in spacetime. When one tackles Special Relativity without geometry, to understand how it is possible that time is not an absolute concept takes much more effort and can be considerably more difficult (at least in the opinion of the relativist that is writing these words).

The fact that Relativity presents this invariance in the choice of coordinates also implies geometry in the behaviour of other physical theories. For example, Classical Electrodynamics, which is inherently relativistic. One can reformulate Electrodynamics in terms of differential forms, which are one of the many tools that geometry provides, and obtain a deeper understanding of the theory. In fact, it allows one to clearly see how it can be generalized to more general theories, such as the Yang–Mills theories that have thrived in High Energy Physics, or even how Electrodynamics could behave in more general spacetimes.

Thermodynamics

Two

Topology

OUR goal will then be to pursue a way of giving proper meaning to formulating physical theories without the need to directly refer to coordinates. Our first step should then be to invent a stage in which we could have success in doing this. Given that we desire to eventually be able to take derivatives, write differential equations, integrate them, and do Calculus in general, we should certainly be able to take limits. Hence, our first step is to get a better grasp at taking limits. In more technical language, we begin by studying Topology. A few good references on the topics covered in here can be found in Folland 1999; Lima 2017; Munkres 2000; Simon 2015; Wald 1984.

2.1 Metric Spaces

To take a limit means to study how a function behaves as its argument gets closer and closer to a point. When doing Calculus in the real line, one writes

$$\lim_{x \to a} f(x) = L \tag{2.1}$$

whenever it holds that

$$\forall\, \epsilon > 0, \exists\, \delta > 0; |x - a| < \delta \Rightarrow |f(x) - L|, \tag{2.2}$$

id est, we can get $f(x)$ as close to the value L as we want, as long as we put x as close to a as we need.

What is interesting about this definition is that what we really need from it is not a fundamental part of the real numbers, but rather to have a notion of "closeness". All the content in this construction depends only on what we mean by being close, and we could exploit this fact to make the notion of limit much broader.

This will be our motivation to define what is a metric, which is the technical term we use to refer to a "distance function", *id est*, a function that measures the distances between points. We begin by stating our desiderata — our list of desired properties. What do we expect from a metric? What should look like a metric? We start by answering this questions, and ultimately choose to keep the properties that end up leading us to useful results.

Let us then pick M to be a set. A metric should be a function $d\colon M \times M \to \mathbb{R}$, which picks two points in M and attribute to them their distance. Two points should have zero distance if, and only if, they are both equal, so we ask that $d(x, y) = 0 \Leftrightarrow x = y$. Furthermore, we would like the metric to resemble the sorts of things we have with our usual notion of Euclidean distance. For example, that any side of a triangle is smaller than or equal to the sum of the remaining sides. We impose this be requiring that $d(x, y) \leq d(x, z) + d(y, z)$: the distance from x to y is never greater than the distance from x to a "detour" z plus the distance from y to the same detour z. These two conditions are in fact sufficient to get a quite reasonable notion of metric.

Definition 2.1 [Metric Space]**:**
Let M be a set and $d\colon M \times M \to \mathbb{R}$ be a function. d is said to be a *metric* in M and (M, d) is said to be a *metric space* if, and only if, the conditions

i. $\forall\, x, y \in M, d(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles);

ii. $\forall\, x, y, z \in M, d(x, y) \leq d(x, z) + d(y, z)$ (triangle inequality),

hold. ♠

We may then prove our first result, which shows these two conditions yield two more particularly interesting results.

Proposition 2.2:
Let (M, d) be a metric space. Then it holds that

i. $\forall\, x, y \in M, d(x, y) = d(y, x)$ *(symmetry);*

ii. $\forall\, x, y \in M, d(x, y) \geq 0$ *(non-negativity).*

□

Proof:
Let $x, y \in M$. From the triangle inequality, we know that

$$d(x, y) \leq d(x, x) + d(y, x), \tag{2.3}$$

and the identity of indiscernibles leads us to

$$d(x, y) \leq d(y, x). \tag{2.4}$$

We can then perform the same argument, but exchanging the positions of x and y at the beginning, to get to $d(y, x) \leq d(x, y)$. Bringing both results together we conclude that $d(x, y) = d(y, x)$.

Next we notice that the triangle inequality also gives us that

$$d(x, x) \leq d(x, y) + d(y, x). \tag{2.5}$$

Using the identity of indiscernibles and the symmetry we just proved we see that

$$0 \leq 2d(x, y), \tag{2.6}$$

and hence $d(x, y) \geq 0$, as desired. ■

With a notion of distance in hands, we can generalize our definition of limits to metric spaces by simply copying what we know from Calculus.

Definition 2.3 [Limits in Metric Spaces]**:**
Let (M, d_M) and (N, d_N) be metric spaces and $f\colon M \to N$ be a function. Given $a \in M$ and $L \in N$, we say that L is the *limit* of $f(x)$ as x tends to a, and write $\lim_{x\to a} f(x) = L$, if, and only if, the expression

$$\forall\, \epsilon > 0, \exists\, \delta > 0; d_M(x, a) < \delta \Rightarrow d_N(f(x), L) < \epsilon \tag{2.7}$$

holds. ♠

Notice this has the exactly same idea as the definition one uses for the real numbers, but we now can use it for far wider contexts. We are now able to take limits in any space that has a metric space structure. Let us then see a few examples of possible metric spaces.

Examples [Taxicab, Euclidean, and Chessboard Metrics]**:**
The first example we should present is $\mathbb{R}^n$ itself endowed with the standard Euclidean metric, d_2, given by

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{2.8}$$

We could also equip $\mathbb{R}^n$ with different metrics, such as

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \tag{2.9}$$

or

$$d_\infty(x, y) = \max_{1\leq i\leq n} |x_i - y_i|. \tag{2.10}$$

The former is sometimes referred to as the taxicab metric. Since, in $\mathbb{R}^2$, it adds the horizontal coordinate to the vertical coordinate, it coincides with the distance a taxicab travels from a point to another in a city with square blocks. The latter metric could be called a chessboard metric, since it measures the distance between different squares in the board — in other words, it gives the number of moves that a king in square x needs to reach square y. ♥

Maybe add a figure

These examples exhibit how different physical situations might lead us to want to employ different notions of distances. Nevertheless, all of them share a few fundamental properties that characterizes them as metrics. Of course, we could drop a few requirements to consider even more general situations, but we are not interested in full generality. Instead, we want to build notions that will be useful to us when studying physical phenomena.

Example [Norms]**:**
Given a normed space $(V, \|\cdot\|)$ — *id est*, a vector space V endowed with a norm $\|\cdot\|$ — we

can get a metric by defining $d(x, y) = \|x - y\|$. Indeed, notice that $d(x, y) = \|x - y\| = 0$ will hold if, and only if, the vector $x - y$ is the null vector. Furthermore, norms are also required to satisfy the triangle inequality. Hence, if we choose a norm in a vector space, we get a metric "for free". As a consequence, we can talk about limits in normed spaces. ♥

This example is widely used in Physics, though sometimes implicitly. For example, we often consider expressions of the form

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |n\rangle \tag{2.11}$$

in Quantum Mechanics. Notice this is just shorthand for

$$|\psi\rangle = \lim_{N\to+\infty} \sum_{n=1}^{N} c_n |n\rangle , \tag{2.12}$$

and the limit makes sense because deep down we have a metric space structure telling us how to take limits.

Example [Function Spaces]**:**
We can generalize the metrics we provided in $\mathbb{R}^n$ to function spaces. For example, let us consider the space $M = \mathcal{C}^0([0, 1], \mathbb{C})$, comprised of the continuous functions $f : [0, 1] \to \mathbb{C}$. We can equip it with the metrics

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x , \tag{2.13}$$

$$d_2(f, g) = \sqrt{\int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x}, \tag{2.14}$$

$$d_\infty(f, g) = \sup_{0\leq x\leq 1} |f(x) - g(x)|. \tag{2.15}$$

Notice d_2 is similar to the metric we use when dealing with wavefunctions in Quantum Mechanics, the difference being essentially in the space of functions we are picking. d_∞ is often called the uniform metric, for it induces the notion of uniform continuity. ♥

Speaking of continuity, we can define continuous functions in a manner analogous to what we have in the real line.

Definition 2.4 [Continuous Functions Between Metric Spaces]**:**
Let (M, d_M) and (N, d_N) be metric spaces and $f : M \to N$ be a function. We say f is *continuous at* $a \in M$ if, and only if, $\lim_{x\to a} f(x) = f(a)$. If f is continuous at $a \in M$ for every $a \in M$ we say f is *continuous*. ♠

We see then that, with a metric, we have all the technology we need to discuss limits and continuities on spaces more abstract than the real line, or even on the real line itself. We should point it out, though, that different metrics can lead to different notions of limits and of continuity on the very same set. See, *exempli gratia*, Problem 2.3 on page 8.

Do we need a metric?

It is interesting to wonder whether the metric space structure is the most fundamental one we can consider to study limits and continuities. In our present formulation, it is hard to imagine how one could formulate the theory in more generality, so we'll repeat our previous procedure: we'll figure out a way of writing the same definitions in another manner and then use this other manner to obtain a more general notion.

So far, we've been explicitly mentioning the distances between points in our definitions of limits. However, these expressions appear usually in the form $d(x, y) < \epsilon$, *id est*, we are not needing the explicit distance between $d(x, y)$, but rather we are bounding what is the maximum allowed distance. This is quite common in the area of Mathematics known as Analysis: one often works with estimates and bounds rather than equations.

A consequence of this fact is that instead of writing $d(x, y) < \epsilon$, we could say that y is inside a ball centered at x with radius ϵ. The idea is the very same one would have in Euclidean geometry: a sphere of radius r centered at p is the set of points at a fixed distance r from p and a ball of radius r centered at p is "the bulk" of the sphere with radius r centered at p.

Definition 2.5 [Open and Closed Balls]**:**
Let (M, d) be a metric space, $p \in M$ and $r > 0$. We define the *open ball* centered at p with radius r to be the set

$$\mathcal{B}_r(p) = \{q \in M; d(p, q) < r\}. \tag{2.16}$$

The *closed ball* $\overline{\mathcal{B}_r(p)}$ is defined similarly, with $<$ replaced by $\leq$. The metric space with respect to which the open (closed) ball is meant is usually understood by context. ♠

With this small change of notation, we may rephrase our notion of limits in metric spaces.

Proposition 2.6 [Limits in Metric Spaces]**:**
Let (M, d_M) and (N, d_N) be metric spaces and $f : M \to N$ be a function. Given $a \in M$ and $L \in N$, $\lim_{x \to a} f(x) = L$, if, and only if, the expression

$$\forall \epsilon > 0, \exists \delta > 0; x \in \mathcal{B}_\delta(a) \Rightarrow f(x) \in \mathcal{B}_\epsilon(L) \tag{2.17}$$

holds. □

There is no new information in Proposition 2.6. It simply rephrased our old definition in terms of new notation. However, it does hint at a new way of thinking about limits: instead of specifying distances with numbers only, we may also try specifying distances with sets.

Our goal then will be to drop the need to directly work with open balls and instead work with a wider collection of sets. Firstly, we'll begin with some definitions.

Definition 2.7 [Interior Point]**:**
Let (M, d) be a metric space and $O \subseteq M$. $p \in O$ is said to be an *interior point* of O if, and only if, there is some $\epsilon > 0$ such that $\mathcal{B}_\epsilon(p) \subseteq O$. ♠

Intuitively, we say a point in a set is an interior point when it is not "on the edge" of the set. There is still some space around it, so it is safely inside the set.

Definition 2.8 [Open Sets]**:**

Let (M, d) be a metric space. A set $O \subseteq M$ is said to be *open* if, and only if, all of its points are interior points. ♠

Essentially, the goal is that the open set is "edgeless". There is always some more points which are closer to the edge without none of them ever being there. The standard example are the open intervals and the open balls. See Problem 2.5.

2.A Problems

Problem 2.1:

Show that the metrics d_1, d_2, and d_∞ — the taxicab, Euclidean, and chessboard metrics — defined on $\mathbb{R}^n$ by Eqs. (2.8) to (2.10) on page 5 are indeed metrics. ✠

Problem 2.2:

In Eq. (2.12) on page 6, we wrote the limit of a sequence, but previously we had defined limits in metric spaces only when considering a point tending towards another point. Given a metric space (M, d) and a sequence $x_n \in M$, define the meaning of the expression $\lim_{n\to\infty} x_n = x$ for some $x \in M$. ✠

Problem 2.3:

Consider the real line $\mathbb{R}$. Let $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases} \tag{2.18}$$

Show that $(\mathbb{R}, d)$ is a metric space. Does the limit $\lim_{n\to+\infty} \frac{1}{n}$ exist in $(\mathbb{R}, d)$? What if, instead of d, we considered the Euclidean metric?

d is sometimes called the discrete metric. ✠

Problem 2.4:

On $\mathbb{R}^2$, draw the open balls centered at the origin with radius 1 for the taxicab, Euclidean, and chessboard metrics. Given an arbitrary, non-empty, set M, write the general expression for an open ball of arbitrary radius centered at an arbitrary point. ✠

Problem 2.5:

Consider $\mathbb{R}$ with the Euclidean metric. Show that sets of the form (a, b), $a < b$, are open.

More generally, consider a metric space and show that all open balls are open. ✠

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