Flat Spacetimes

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Abstract: These are some examples of how some quite interesting results in Relativity can be found without ever leaving Minkowski spacetime. We consider some universes within Minkowski spacetime—*viz.* the Rindler and Milne universes—and use them to understand some properties of black holes and cosmology. Our main references are the books by Ellis and Williams (2000) and Wald (1984).

Keywords: General Relativity, Special Relativity, Minkowski spacetime, Rindler universe, Milne universe.

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1 Introduction

In this text, we'll be working with the Minkowski spacetime $(\mathbb{R}^2, \eta_{ab})$, where the metric η_{ab} is given in inertial, Cartesian coordinates by

$$ds^2 = -dt^2 + dx^2 . \tag{1.1}$$

The choice of working in only two coordinates is to simplify the computations, but many of our results can be generalized to more dimensions in a straightforward manner.

The notation and conventions here employed follow closely those used by Wald (1984), which corresponds to the $+\,+\,+$ sign convention in the Misner, Thorne, and Wheeler (1973) classification. As for the indices, we employ abstract index notation (see Geroch 2013; Wald 1984): Latin indices such as a, b, c, etc are used to represent tensors themselves and Greek indices such as μ, ν, ρ, etc to indicate components with respect to some choice of coordinates. When, and if necessary, we'll explicitly indicate whether the Latin indices correspond to spatial tensor, in which case we'll usually employ the letters i, j, k, etc.

2 Rindler Universe

2.1 Accelerated Observers

Typically, when considering the Minkowski spacetime, one understands time as given by the vector $t^a \equiv \left(\frac{\partial}{\partial t}\right)^a$ (t being the inertial coordinate that occurs on Eq. (1.1) on the preceding page). This corresponds to the proper time of inertial observers moving through spacetime four-velocity given by $u^a = t^a$. The time parameter t can be recovered from this vector field by imposing that it be the function on spacetime such that $t^a \nabla_a t = 1$.

One of the main properties of this vector field is that it is one of the Killing fields of Minkowski spacetime, *i.e.*, the one-parameter group generated by this vector field corresponds to an isometry of the Minkowski metric (see, *e.g.*, Wald 1984, App. C). In other words, it is one of the generators of the Poincaré group, being responsible for generating the time-translations of Minkowski spacetime.

However, one can often pick other generators of the Poincaré group as a "notion of time". Let us consider the vector field given by

$$b^a = \kappa\left(x\left(\frac{\partial}{\partial t}\right)^a + t\left(\frac{\partial}{\partial x}\right)^a\right), \tag{2.1}$$

where $\kappa > 0$ is a constant. One can show that this is also a Killing field for the Minkowski metric. Let us figure out its physical meaning by considering which sorts of curves run parallel to it, *i.e.*, we want to consider which curves have a tangent b^a. In inertial coordinates, we are trying to solve

$$\begin{cases} \dot{t} = \kappa x, \\ \dot{x} = \kappa t, \end{cases} \tag{2.2}$$

where the dots denote differentiation with respect to a parameter. Solving the system of differential equations we find that the curves are given by

$$\begin{cases} t(\tau) = t_0 \cosh(\kappa\tau) + x_0 \sinh(\kappa\tau), \\ x(\tau) = t_0 \sinh(\kappa\tau) + x_0 \cosh(\kappa\tau), \end{cases} \tag{2.3}$$

where $t(0) = t_0$ and $x(0) = x_0$. Despite the weird form, one can recognize that the transformation in Eq. (2.3) corresponds to a Lorentz boost with rapidity $w = -\kappa\tau$. Hence, b^a is a generator of boosts on Minkowski spacetime.

While it is interesting that b^a generates a symmetry of Minkowski spacetime, this still doesn't give us much insight on which observers have four-velocity b^a. Let us begin by figuring out in which regions of Minkowski spacetime b^a can even be understood as a four-velocity, *i.e.*, in which regions it is timelike. We can see from Eq. (2.1) that

$$b^a b_a = \kappa^2\left(-x^2 + t^2\right). \tag{2.4}$$

From this expression, we see that the behavior of b^a as a timelike, null, or spacelike field depends on whether $|x|$ is larger or smaller than $|t|$.

It is easier to perceive these relations by plotting the flow parallel to b^a, as done on Fig. 2.1 on the next page. We can then see that the path parallel to b^a is timelike on the left and right Rindler

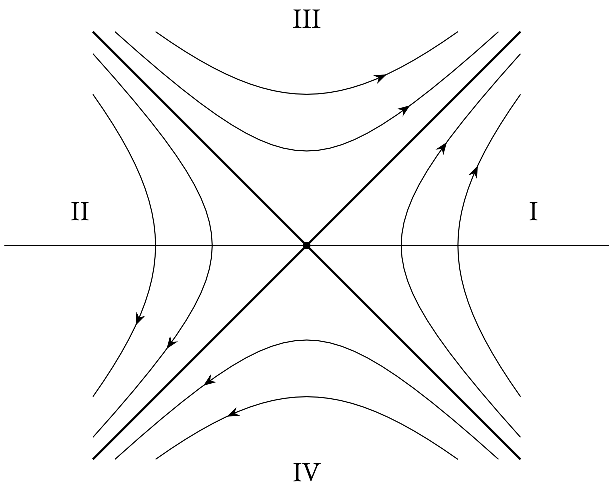

Figure 2.1: *Rindler wedges of Minkowski spacetime. The flow of the vector field defined on Eq.* (2.1) *on the previous page is shown by the lines with arrows. We can see that b^a is timelike on the left and right wedges, I and II, but spacelike on the future and past wedges, III and IV. The vertical axis corresponds to the t coordinate and the horizontal axis to the x coordinate.*

wedges (the regions with $-x > |t|$ and $x > |t|$, respectively), but spacelike on the future and past Rindler wedges ($t > |x|$ and $-t > |x|$).

Hence, we can understand b^a as providing a notion of time on the right and left Rindler wedges, but we still are not sure of what observers have this notion of proper time. To figure that out, let us impose that $b^a b_a = -1$, which tells us with some aid of Eq. (2.4) on the preceding page that such an observer follows a trajectory respecting

$$x = \pm\sqrt{\frac{1}{\kappa^2} + t^2}. \tag{2.5}$$

Let us stick to the right Rindler wedge and pick the upper sign. With this relation in mind, let us compute the four acceleration of this observer. It is given by

$$\begin{aligned} a^a &= b^b \nabla_b b^a, && (2.6a)\\ a^t &= b^\mu \nabla_\mu b^t, && (2.6b)\\ &= \kappa^2 \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \right) x, && (2.6c)\\ &= \kappa^2 t, && (2.6d)\\ a^x &= \kappa^2 x, && (2.6e) \end{aligned}$$

where the computation of a^x goes just as that for a^t. Hence, we can see that

$$\begin{aligned} a_a a^a &= \kappa^4 \eta_{ab} \left(t\left(\frac{\partial}{\partial t}\right)^a + x\left(\frac{\partial}{\partial x}\right)^a \right)\left(t\left(\frac{\partial}{\partial t}\right)^b + x\left(\frac{\partial}{\partial x}\right)^b \right), && (2.7a)\\ &= \kappa^4\left(-t^2 + x^2\right), && (2.7b)\\ &= \kappa^2, && (2.7c) \end{aligned}$$

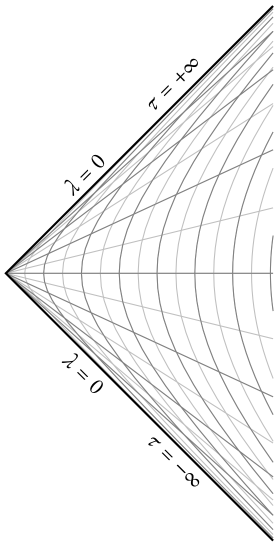

Figure 2.2: *Depiction of how the Rindler coordinates cover the right Rindler wedge. The thick black lines at $\lambda = 0$, $\tau = \pm\infty$ correspond to the lines $|x| = |t|$ in inertial coordinates. The hyperbolae are curves of constant λ, with larger values of λ the further away the hyperbola is from the horizon. The straight gray lines are curves of constant τ, with lines with larger positive slope corresponding to larger values of τ.*

which means the acceleration of the observer following the path described by b^a is $\sqrt{a^a a_a} = \kappa$. Hence, b^a corresponds to the four-velocity of an observer moving through spacetime with constant acceleration κ.

2.2 Rindler Coordinates

Let us try to find coordinates on the right Rindler wedge that explicit use the proper time of accelerated observers as a coordinate. To do so, we can attempt at using as coordinate transformation the expressions for Eq. (2.3) on page 2 with $t_0 = 0$ and x_0 turned into a new variable, say λ. That is to say we'll consider the coordinate transformation given by

$$\begin{cases} t \to \lambda \sinh(\kappa\tau), \\ x \to \lambda \cosh(\kappa\tau), \end{cases} \tag{2.8}$$

for $\lambda > 0$ (we're covering only the right wedge) and $\tau \in \mathbb{R}$. Notice the inverse transformation is

$$\begin{cases} \lambda \to \sqrt{x^2 - t^2}, \\ \tau \to \dfrac{1}{\kappa} \operatorname{artanh}\left(\dfrac{t}{x}\right). \end{cases} \tag{2.9}$$

In these coordinates, the right Rindler wedge looks like Fig. 2.2.

After a little manipulation one can see that the Minkowski metric is now written as

$$\mathrm{d}s^2 = -\kappa^2\lambda^2\,\mathrm{d}\tau^2 + \mathrm{d}\lambda^2\,. \tag{2.10}$$

Notice this metric can appear problematic at $\lambda = 0$, since one of the components vanishes. While this could seem like a problem if we knew only the Rindler coordinates, we can see from Eq. (2.9) on the previous page that $\lambda = 0$ means simply that $|x| = |t|$ in inertial coordinates. It is clearly simply a feature of our choice of a coordinates, not an actual problem with spacetime itself.

2.3 Geodesics

Let us pretend we did not know that we are in Minkowski spacetime and were given the metric Eq. (2.10) on the preceding page and asked to understand what is going on in this universe. Let us being by finding the geodesics. Since $b^a = \left(\frac{\partial}{\partial\tau}\right)^a$ is a Killing field, it holds that the quantity

$$u^a b_a = -E \tag{2.11}$$

is conserved throughout a geodesic with (affinely-parameterized) tangent u^a. Opening up in components, we find that

$$\kappa^2\lambda^2\dot{\tau} = E \tag{2.12}$$

is constant, where the dot denotes differentiation with respect to the geodesic's affine parameter (which we'll denote ζ for the time being).

From the metric of Eq. (2.10) on the previous page we get that

$$-\kappa^2\lambda^2\dot{\tau}^2 + \dot{\lambda}^2 = -k, \tag{2.13}$$

where

$$k = \begin{cases} +1, \text{for timelike geodesics,} \\ 0, \text{for null geodesics.} \end{cases} \tag{2.14}$$

Hence, one gets to the system of differential equations

$$\begin{cases} \dot{\lambda}^2 + k - \dfrac{E^2}{\kappa^2\lambda^2} = 0, \\ \dot{\tau} = \dfrac{E}{\kappa^2\lambda^2}, \end{cases} \tag{2.15}$$

which can be solved quite easily (for example, with `Mathematica`). For $k = 1$ (timelike geodesics) the solution is

$$\begin{cases} \lambda(\zeta) = \sqrt{\dfrac{E^2}{\kappa^2} - (\zeta + C_1)^2}, \\ \tau(\zeta) = \dfrac{1}{\kappa}\operatorname{artanh}\left(\dfrac{\kappa(\zeta + C_1)}{E}\right) + C_2, \end{cases} \tag{2.16}$$

where C_1 and C_2 are integration constants. As for $k = 0$ (null geodesics), the solution is

$$\begin{cases} \lambda(\zeta) = \sqrt{\dfrac{2E(\pm\zeta + C_1)}{\kappa}}, \\ \tau(\zeta) = \pm\dfrac{1}{2\kappa}\log\left(2\kappa(\pm\zeta + C_1)\right) + C_2, \end{cases} \tag{2.17}$$

and once again C_1 and C_2 are integration constants and the choice of sign must be the same in both expressions. Notice that in all of these expressions the constant C_1 is just a translation on the parameter ζ, which is an irrelevant freedom for our purposes. Hence, we can write the previous solutions as

$$\begin{cases} \lambda(\zeta) = \sqrt{\frac{E^2}{\kappa^2} - \zeta^2}, \\ \tau(\zeta) = \frac{1}{\kappa} \operatorname{artanh}\left(\frac{\kappa\zeta}{E}\right) + C, \end{cases} \tag{2.18}$$

for the timelike geodesics and as

$$\begin{cases} \lambda(\zeta) = \sqrt{\pm\frac{2E\zeta}{\kappa}}, \\ \tau(\zeta) = \pm\frac{1}{2\kappa} \log(\pm 2\kappa\zeta) + C, \end{cases} \tag{2.19}$$

for the null geodesics.

Timelike Geodesics

Let us first focus on the timelike geodesics. We begin by noticing from Eq. (2.18) that at proper time $\zeta = -\frac{E}{\kappa}$ the particle emerges from the horizon at $\lambda = 0$ and at $\zeta = +\frac{E}{\kappa}$ the particle vanishes again on the horizon. Notice this happens in finite proper time.

Let us now figure out for how long the accelerated observers see the particle after it emerges from the horizon and before it vanishes behind it again. The particle emerges from the horizon at $\zeta = -\frac{E}{\kappa}$, which according to Eq. (2.18) corresponds to

$$\tau\left(-\frac{E}{\kappa}\right) = \frac{1}{\kappa} \operatorname{artanh}(-1) + C \to -\infty, \tag{2.20}$$

which means the accelerated observer never saw the particle emerge from the horizon: it came from there in the far past. Similarly, the time at which the particle vanishes again behind the horizon is, according to the accelerated observer,

$$\tau\left(+\frac{E}{\kappa}\right) = \frac{1}{\kappa} \operatorname{artanh}(+1) + C \to +\infty, \tag{2.21}$$

and hence it takes infinite time for the particle to cross the horizon.

While this might seem odd from the point of view of the accelerated observer, it is not strange from the point of view of an inertial observer—we never left Minkowski spacetime and the geodesics are simply straight lines through spacetime. Crossing the horizon means simply crossing the lines $|x| = |t|$, and there is nothing particularly special about that. Nevertheless, the choice of Rindler coordinates makes it seem like there is a big deal happening in there.

For completeness, we notice that by manipulating Eq. (2.18) we can write the position of the particle in the λ coordinate as a function of the time τ as

$$\lambda(\tau) = \frac{E}{\kappa} \operatorname{sech}\left(\kappa(\tau - C)\right). \tag{2.22}$$

Null Geodesics

As for the null geodesics, let us focus on a different aspect: what is the speed of light as perceived by the accelerated observers?

Since light travels by null geodesics, what we want to compute is simply $\frac{d\lambda}{d\tau}$ along a null geodesic. In order to do this, we can use Eq. (2.15) on page 5 instead of the solutions to the equations of motion. From Eq. (2.15) on page 5 with $k = 0$ we can see that

$$\left(\frac{d\lambda}{d\tau}\right)^2 = \frac{\dot{\lambda}^2}{\dot{\tau}^2}, \tag{2.23a}$$

$$= \frac{E^2}{\kappa^2\lambda^2}\frac{\kappa^4\lambda^4}{E^2}, \tag{2.23b}$$

$$= \kappa^2\lambda^2, \tag{2.23c}$$

$$\frac{d\lambda}{d\tau} = \pm\kappa\lambda, \tag{2.23d}$$

where the sign simply refers to the direction of propagation. We notice, however, that the speed of light needs not to be one, since λ is not fixed at $\pm\frac{1}{\kappa}$.

Let's figure out what is the precise expression of the speed of light as a function of τ. Solving for ζ and substituting in the expression for τ we get to

$$\tau = \pm\frac{1}{2\kappa}\log\left(\frac{\kappa^2\lambda^2}{E}\right) + C, \tag{2.24}$$

and then solving for λ leads to

$$\lambda(\tau) = \frac{\sqrt{E}}{\kappa}e^{\pm\kappa(\tau-C)}, \tag{2.25}$$

from which we see that, for an accelerated observer, not only the speed of light needs not to be 1, but it doesn't even need to be constant. This, of course, is not in conflict with Einstein's second postulate of Special Relativity, which states simply that the speed of light is the same in all *inertial* frames of reference.

2.4 Redshift at the Horizon

Let us assume a beacon is sent by some observer to the horizon. The beacon sends light signals with a frequency ω_0 and has a rocket system such that, after moving for some time, allows it to sit at a constant $\lambda = \lambda_0$. Assuming the observer is fixed at λ, what is the frequency the observer measures for the signals emitted by the beacon?

Both the beacon and the observer have four-velocities given by

$$u^a = \frac{b^a}{\sqrt{-b^b b_b}} \tag{2.26}$$

despite at different points. Suppose the light signal has four-momentum k^a, so that ($\hbar = 1$)

$$\omega_0 = -\left.k^a u_a\right|_{\lambda_0}. \tag{2.27}$$

The frequency measured by the observer will be

$$\omega = - \left. k^a u_a \right|_\lambda . \tag{2.28}$$

Recalling that $k^a b_a$ is a constant (because k^a satisfies the geodesic equation and b^a is a Killing field), we see that

$$\frac{\omega_0}{\omega} = \frac{\left. k^a u_a \right|_{\lambda_0}}{\left. k^a u_a \right|_\lambda}, \tag{2.29a}$$

$$= \frac{\left. k^a b_a \right|_{\lambda_0}}{\left. \sqrt{-b^b b_b} \right|_{\lambda_0}} \frac{\left. \sqrt{-b^b b_b} \right|_\lambda}{\left. k^a b_a \right|_\lambda}, \tag{2.29b}$$

$$= \frac{\left. \sqrt{-b^b b_b} \right|_\lambda}{\left. \sqrt{-b^b b_b} \right|_{\lambda_0}}. \tag{2.29c}$$

Since $b^a = \left(\frac{\partial}{\partial \tau}\right)^a$, we see from Eq. (2.10) on page 4 that $b^a b_a = -\kappa^2 \lambda^2$, which leads us to

$$\frac{\omega_0}{\omega} = \frac{\lambda}{\lambda_0}, \tag{2.30}$$

i.e.,

$$\omega = \frac{\lambda_0}{\lambda} \omega_0 < \omega_0. \tag{2.31}$$

Notice that the closer the beacon gets to the horizon at $\lambda_0 = 0$, the smaller the frequency measured by the observer, *i.e.*, the larger the redshift.

Let us now assume that the beacon is moving inertially. In this case, we have (denoting $l^a = \left(\frac{\partial}{\partial \lambda}\right)^a$ and writing v^a for the beacon's four-velocity)

$$\omega_0 = - \left. k_a v^a \right|_{\lambda_0} = -\dot{\tau} \left. k_a b^a \right|_{\lambda_0} - \dot{\lambda} \left. k_a l^a \right|_{\lambda_0} . \tag{2.32}$$

Let us write $k^a = k^\tau b^a + k^\lambda l^a$. Since $k^a k_a = 0$, we have at λ_0

$$-\kappa^2 \lambda_0^2 k^{\tau 2} + k^{\lambda 2} = 0, \tag{2.33}$$

and hence

$$k^\lambda = \kappa \lambda_0 k^\tau, \tag{2.34}$$

the choice of sign being so that the light ray is future-directed and moves towards increasing values of λ. Notice, however, that $\left. k_a l^a \right|_{\lambda_0} = k^\lambda$ and $\left. k_a b^a \right|_{\lambda_0} = -\kappa^2 \lambda_0^2 k^\tau$. Hence, we managed to show that

$$\left. k_a l^a \right|_{\lambda_0} = -\frac{1}{\kappa \lambda_0} \left. k_a b^a \right|_{\lambda_0} . \tag{2.35}$$

As a consequence, we now know that

$$\omega_0 = -\left(\dot{\tau} - \frac{\dot{\lambda}}{\kappa\lambda_0}\right) k_a b^a\Big|_{\lambda_0}. \tag{2.36}$$

From Eq. (2.15) on page 5 we see that

$$\omega_0 = -\left(\frac{E}{\kappa^2\lambda_0^2} + \frac{1}{\kappa\lambda_0}\sqrt{\frac{E^2}{\kappa^2\lambda_0^2} - 1}\right) k_a b^a\Big|_{\lambda_0}, \tag{2.37}$$

where we chose $\dot{\lambda} < 0$ so that the particle is moving towards the horizon. We still need to fix the constant E (which corresponds to choosing the initial conditions for the particle), but we'll do it later. For now, let us recall that the frequency measured by the observer is

$$\omega = -k_a u^a\Big|_{\lambda} = -\frac{k_a b^a\big|_{\lambda}}{\sqrt{b_b b^b\big|_{\lambda}}} = -\frac{k_a b^a\big|_{\lambda_0}}{\sqrt{b_b b^b\big|_{\lambda}}}, \tag{2.38}$$

which allows us to rewrite

$$\omega_0 = \left(\frac{E}{\kappa^2\lambda_0^2} + \frac{1}{\kappa\lambda_0}\sqrt{\frac{E^2}{\kappa^2\lambda_0^2} - 1}\right)\sqrt{b_b b^b\Big|_{\lambda}}\,\omega. \tag{2.39}$$

We already know that $\sqrt{b_b b^b\big|_{\lambda}} = \kappa\lambda$, and hence

$$\omega_0 = \kappa\lambda\left(\frac{E}{\kappa^2\lambda_0^2} + \frac{1}{\kappa\lambda_0}\sqrt{\frac{E^2}{\kappa^2\lambda_0^2} - 1}\right)\omega. \tag{2.40}$$

We can't proceed further without specifying E, so let us assume that the observer dropped the beacon from rest with respect to the observer at $\tau = 0$. Imposing this on Eq. (2.22) on page 6 leads to $E = \kappa\lambda$, and hence the redshift will be such that

$$\omega_0 = \frac{\lambda}{\lambda_0}\left(\frac{\lambda}{\lambda_0} + \sqrt{\frac{\lambda^2}{\lambda_0^2} - 1}\right)\omega. \tag{2.41}$$

This time the redshift depends on time, but we can use Eq. (2.22) on page 6 to describe it. The beacon moves in the trajectory

$$\lambda_0(\tau) = \lambda\,\mathrm{sech}(\kappa\tau), \tag{2.42}$$

where λ is the observer's position and τ is the Rindler coordinate. Rearranging our previous expression leads us to

$$\omega = e^{-\kappa\tau}\,\mathrm{sech}(\kappa\tau)\omega_0, \quad \text{for } \tau > 0, \tag{2.43}$$

where the condition on τ is due to manipulation of the square root (also, we assumed the beacon is dropped at $\tau = 0$, so whatever happens before that is not of our interest). Hence, we see that there is an exponential redshift of the signal of the beacon.

Combining this with our previous computation that is takes infinite time for the beacon to cross the horizon, we see that the accelerated observer will see the beacon approaching the horizon asymptotically and, as it does, they will see the beacon becoming redder and redder due to the redshift effects. While they will never see the beacon crossing the horizon, the redshift will eventually be so large that the light emitted or reflected by the beacon will be too far on the infrared to be visible to the naked eye and the observer will no longer see the beacon.

2.5 Crossing the Horizon

Let us make a change of coordinates on the Rindler universe so we can explore a bit beyond the horizon. Let us define a new coordinate $\rho = \frac{\lambda^2}{4}$. Using this definition, the metric on Eq. (2.10) on page 4 becomes

$$\mathrm{d}s^2 = -4\kappa^2 \rho \,\mathrm{d}\tau^2 + \frac{1}{\rho}\,\mathrm{d}\rho^2 . \tag{2.44}$$

By rescaling τ we can redefine the coordinates so that the metric reads

$$\mathrm{d}s^2 = -\rho \,\mathrm{d}\tau^2 + \frac{1}{\rho}\,\mathrm{d}\rho^2 \tag{2.45}$$

We know the Rindler universe corresponds to the region $\rho > 0$. The region with $\rho = 0$ now appears singular, but since we never really left Minkowski spacetime we know this is just a coordinate singularity. A trick for us to get rid of the coordinate singularity is to employ Eddington–Finkelstein-like coordinates: we compute the spacetime's null geodesics and see how we can index each incoming or outgoing geodesic by a number, and then employ these numbers as coordinates.

First we begin by computing the null geodesics. Imposing $\mathrm{d}s^2 = 0$ on Eq. (2.45) leads us to

$$\rho \,\mathrm{d}\tau^2 = \frac{1}{\rho}\,\mathrm{d}\rho^2 , \tag{2.46}$$

which we can recast as the differential equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}\rho} = \pm\frac{1}{\rho}, \tag{2.47}$$

where the sign corresponds to whether the light ray is incoming or outgoing, *i.e.*, to whether it approaches or gets away from the horizon. Solving the differential equation leads us to

$$\tau = \pm \log|\rho| + C, \tag{2.48}$$

where C is a constant. Let us consider the case

$$\tau + \log|\rho| = v, \tag{2.49}$$

for constant v. Notice that this is an incoming light ray: as τ grows, $\rho > 0$ must become smaller for v to remain constant. Let us now employ v as a coordinate on spacetime. Since $\mathrm{d}v = \mathrm{d}\tau + \frac{1}{\rho}\,\mathrm{d}\rho$, one has from Eq. (2.45) that the metric can now be written as

$$\mathrm{d}s^2 = -\rho \,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\rho , \tag{2.50}$$

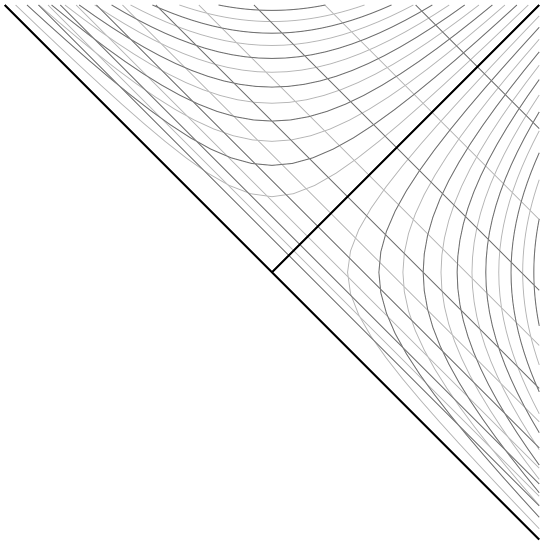

Figure 2.3: *Depiction of how the coordinates of Eq.* (2.50) *on the previous page cover the right and future Rindler wedges. The thick black lines at $\rho = 0$ correspond to the lines $|x| = |t|$ in inertial coordinates, and $v = -\infty$ on $x = -t$. The hyperbolae are curves of constant ρ, with larger values of λ the further away the hyperbola is from the horizon for the right Rindler wedge, and more negative values the further away the hyperbola is from the horizon for the future Rindler wedge. The straight gray lines are curves of constant v. The expressions for inertial coordinates of Minkowski spacetime in terms of the coordinates of Eq.* (2.50) *on the preceding page that allow for this diagram to be drawn are computed on Section 2.6.*

which is no longer singular at $\rho = 0$. In fact, now the coordinates $\rho < 0$ are perfectly valid, and hence we can extend the Rindler universe to comprehend the region $\rho < 0$ as well. With these considerations, it can be shown (see Section 2.6) that the region of Minkowski spacetime covered by these coordinates ranging from $-\infty$ to $+\infty$ is the one shown in Fig. 2.3, which corresponds to the right and future Rindler wedges.

Let us compute the geodesics in these new coordinates. Firstly, we compute the null geodesics. They respect

$$-\rho\left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^2 + 2\frac{\mathrm{d}v}{\mathrm{d}\zeta}\frac{\mathrm{d}\rho}{\mathrm{d}\zeta} = 0, \tag{2.51}$$

where ζ is an affine parameter. There are two possibilities—namely,

$$v = C \quad \text{or} \quad \rho = Ce^{\frac{v}{2}}, \tag{2.52}$$

where C is a constant.

As for the timelike geodesics, the Lagrangian for the metric is

$$L = -\frac{\rho\dot{v}^2}{2} + \dot{v}\dot{\rho}, \tag{2.53}$$

and hence the Euler–Lagrange equations read

$$\begin{cases} \ddot{v} + \dfrac{\dot{v}^2}{2} = 0, \\ \dot{\rho} - \rho\dot{v} = -C, \end{cases} \tag{2.54}$$

for some integration constant C. Imposing on the metric that the solution is timelike gives us that

$$-\rho\left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^2 + 2\frac{\mathrm{d}v}{\mathrm{d}\zeta}\frac{\mathrm{d}\rho}{\mathrm{d}\zeta} = -1, \tag{2.55a}$$

$$-\rho\left(\frac{C}{\rho} + \frac{\dot{\rho}}{\rho}\right)^2 + 2\frac{C\dot{\rho}}{\rho} + \frac{2\dot{\rho}^2}{\rho} = -1. \tag{2.55b}$$

Rearranging the expression we get to

$$\dot{\rho}^2 = C^2 - \rho. \tag{2.56}$$

The differential equation is separable, and its solution is given by

$$\rho(\zeta) = C^2 - \frac{1}{4}(\zeta + C_1)^2, \tag{2.57}$$

for any constant C_1. Since C_1 simply shifts the ζ variable, we'll take it to be zero for simplicity. As for v, it is given by

$$\dot{v} = \frac{C - \frac{1}{2}\zeta}{C^2 - \frac{1}{4}\zeta^2} = C + \frac{1}{2}\zeta. \tag{2.58}$$

Integrating, we get to

$$v(\zeta) = 2\log|2C + \zeta| + C_2, \tag{2.59}$$

for some integration constant C_2.

From Eq. (2.57) we can see that, in a finite time proper time, the particle emerges from $\rho \leq 0$ and in a finite proper time it dives back to $\rho \leq 0$. Even though the particle spends an infinite amount of time in the region with $\rho > 0$ when measured by an accelerating observer, in the particle's reference frame this happens in a finite amount of time. If we employ the expression for $v(\zeta)$, we find that the motion of the particle is described in (v, ρ) coordinates as

$$\rho(v) = -\frac{1}{4}e^v + C_1 e^{\frac{v+C_2}{2}}, \tag{2.60}$$

where the constants C_1 and C_2 are arbitrary. In this coordinates, the particle crossed the horizon in the infinitely far past, spends an infinite amount of time in the $\rho > 0$ region, but in a finite time it crosses back to $\rho \leq 0$.

2.6 Recovering Minkowski Spacetime

Let us start from the metric on Eq. (2.45) on page 10 and work our way back to Minkowski coordinates. This will also allow us to get the expressions needed to draw Fig. 2.3 on the previous page, and is an exercise on how to get to the so-called Kruskal extension of the Schwarzschild spacetime.

Previously we found that in the (τ, ρ) coordinates the null geodesics could be written as in Eq. (2.48) on page 10. We then proceeded to define the new coordinate

$$v = \tau + \log|\rho|, \tag{2.61}$$

which we now supplement with the definition

$$u = \tau - \log|\rho|. \tag{2.62}$$

The inverse transformations are given by

$$\tau = \frac{u+v}{2} \quad \text{and} \quad \rho = \pm e^{\frac{v-u}{2}}. \tag{2.63}$$

Using these expressions, we can show that Fig. 2.3 on page 11 can be rewritten as

$$ds^2 = \mp e^{\frac{v-u}{2}} \, du \, dv, \tag{2.64}$$

where the upper sign refers to the region with $\rho > 0$ and the lower sign to the region with $\rho < 0$. Hence, this choice of coordinates still does not allow us to analyze what is going on at the apparently problematic region with $\rho = 0$.

Let us try to remove this factor from the front of $du\,dv$. In order to do so, let us notice that in the metric given in the form of Eq. (2.45) on page 10, $\left(\frac{\partial}{\partial\tau}\right)^a$ is a Killing field. Hence, if k^a is tangent to a null geodesic, the quantity

$$E = -k_a \left(\frac{\partial}{\partial\tau}\right)^a = \rho \frac{d\tau}{d\lambda} \tag{2.65}$$

is conserved, where λ is an affine parameter along the geodesic. Our point with this is that we can try to solve for the affine parameter and use it as a coordinate, which will lead to a coordinate choice naturally adapted to the null geodesics of the spacetime, and hence simpler than our simple choice of "pick a constant for each geodesic".

We can rewrite the previous equation as

$$d\lambda = \frac{\rho}{E} d\tau, \tag{2.66a}$$

$$= \pm \frac{e^{\frac{v-u}{2}}}{2E} d(u+v). \tag{2.66b}$$

If we keep u constant (*i.e.*, if we consider an outgoing geodesic), integrating the expression will lead us to

$$\lambda = \pm \frac{e^{\frac{v-u}{2}}}{E} + \text{constant}, \tag{2.67}$$

and hence we see that $\pm e^{\frac{v}{2}}$ is an affine parameter along the outgoing null geodesics. Similarly, one finds that $\mp e^{-\frac{u}{2}}$ is an affine parameter along the incoming null geodesics. Hence, we'll define the new coordinates

$$V = e^{\frac{v}{2}} \quad \text{and} \quad U = \mp e^{-\frac{u}{2}}. \tag{2.68}$$

The lack of sign in the front of V is due to a coordinate ambiguity we'll come back to in a few paragraphs.

Discuss this (Minkowski "white hole"). For now, we're considering $V > 0$ only.

Using these new coordinates, Eq. (2.64) on the previous page can be rewritten as

$$ds^2 = -dU\,dV\,. \tag{2.69}$$

If we now define

$$U = T - X \quad \text{and} \quad V = T + X, \tag{2.70}$$

then we get to

$$ds^2 = -dT^2 + dX^2\,, \tag{2.71}$$

which is just the Minkowski metric.

If we go back to Eqs. (2.63) and (2.68) on the preceding page, we can see that

$$\rho = -UV \tag{2.72}$$

and Eq. (2.70) implies

$$\rho = -T^2 + X^2. \tag{2.73}$$

Notice then that the same value of ρ is given to two parts of a hyperbola in the original Minkowski diagram, Fig. 2.1 on page 3. Namely, there is a hyperbola with $\rho = \rho_0 > 0$ on the right Rindler wedge, but a symmetrical hyperbola in the left wedge with the same value. These sorts of ambiguity in the coordinates gained an example in our choices of sign for V and U on Eq. (2.68) on the previous page—notice that the choices of sign correspond to each of the different Rindler wedges:

- right wedge: $U < 0, V > 0$;
- left wedge: $U > 0, V < 0$;
- future wedge: $U > 0, V > 0$;
- past wedge: $U < 0, V < 0$.

Our choice of signs corresponded to the future and right wedges, but we could have chosen different ones. Notice that once we get to the inertial coordinates, we realize there is nothing forbidding us from considering values of V smaller than zero, and similar conclusions would hold for other sign choices. In all possibilities, we eventually arrive at the conclusion that we can consider a larger complete spacetime. The lesson is that some coordinate choices might not cover the whole spacetime, and there's nothing really wrong with it. To deal with these issues, we ask in General Relativity for spacetimes to be inextendible (for more detail, see, *e.g.*, Hawking and Ellis 1973, Sec. 3.1).

Notice also that from Eqs. (2.68) and (2.70) on the preceding page and on the current page we find that

$$v = 2\log|T + X|. \tag{2.74}$$

From Eqs. (2.73) and (2.74) we can draw Fig. 2.3 on page 11 by plotting the curves of constant ρ or v in the future and right wedges (notice we could have also chosen to do a similar figure with the past and left wedges).

2.7 Conclusions

Let us explicitly write some conclusions we can obtain from our analysis of the Rindler universe.

Firstly, we notice how different observers can be associated with different notions of time. While inertial observers in Minkowski spacetime have their four-velocity aligned to some timelike Killing field, accelerated observers follow a different Killing field. We could, of course, consider observers whose four-velocities aren't even Killing fields, although the lack of symmetry in those situations might limit the computations we can make.

Furthermore, we can notice what an important role the coordinates play in our perception of the results we compute. While inertial coordinates make geodesic motion in Minkowski spacetime quite simple, our choice of working in different sets of coordinates made the motion quite complicated and led to weird consequences, such as the appearance of an event horizon in the form of coordinate singularities and infinite redshifts of bodies that took infinitely long to cross the horizon. Of course, by going back to Minkowski coordinates we notice that much of the weirdness was coordinate-induced, and hence does not correspond to anything intrinsically problematic about the spacetime.

While this might seem silly in this context, it is important to pay attention to these issues, since very similar situations occur, for example, when dealing with black holes. In those cases, we often face the coordinates which suggest trouble at a certain value before we find the "nice" coordinates that make the general structure simpler. That might lead to the wrong impression that something extraordinary is happening at a coordinate singularity, but that need not to be the case.

We could also notice how a choice of coordinates might not cover the whole spacetime, but only a portion of it. For a Rindler universe, we managed to find coordinates that could fill the whole Minkowski spacetime. Nevertheless, that is not always the case. For example, it is not possible to cover a sphere with a single continuous coordinate chart[1].

In general, we see that we should be always careful with conclusions that depend too much on coordinates. While they are useful computational tools, different coordinate choices can tell different stories about the physical reality and it is important to watch out for which features of spacetime are due to the manifold itself and which are merely due to the particular coordinate choice one is working on.

3 Milne Universe

3.1 Milne Coordinates

Let us consider the coordinate transformation Eq. (2.8) on page 4 once again, but this time let us try to employ it on the future light cone of the origin[2]. Since we now have $t > |x|$, we'll define the

[1]The reason lies in Topology and is that the sphere and the plane are not homeomorphic. A way of seeing this is by noticing that the sphere is compact, but the plane is not.

[2]In two spacetime dimensions, the future light cone of the origin corresponds to the future Rindler wedge. However, in more dimensions these are different concepts. The future light cone is given by $t > r$ (where r is the radial coordinate), while the future Rindler wedge is given by $t > |x|$ (where x is some Cartesian coordinate). We'll keep our computations in two dimensions for simplicity, but keep in mind that one must be careful when comparing the Rindler and Milne universes in more dimensions.

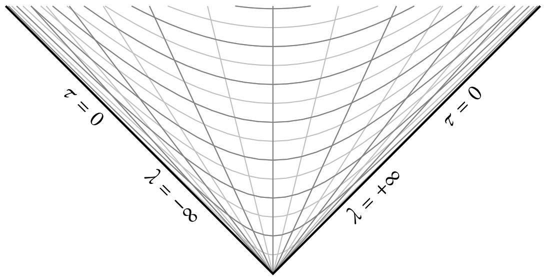

Figure 3.1: *Depiction of how the Milne coordinates cover the future light cone. The thick black lines at $\tau = 0$, $\lambda = \pm\infty$ correspond to the lines $|x| = |t|$ in inertial coordinates. The hyperbolae are spatial curves of constant τ, with larger values of τ the further up the hyperbola is. The straight gray lines are curves of constant λ, with values of λ growing as one moves from $t = -x$ to $t = +x$.*

change of coordinates as

$$\begin{cases} t \to \tau \cosh \lambda, \\ x \to \tau \sinh \lambda, \end{cases} \tag{3.1}$$

for $\tau > 0$ (we're covering only the future wedge) and $\lambda \in \mathbb{R}$. The inverse transformation is

$$\begin{cases} \lambda \to \operatorname{artanh}\left(\frac{x}{t}\right), \\ \tau \to \sqrt{t^2 - x^2}. \end{cases} \tag{3.2}$$

In these coordinates, the future light cone looks like Fig. 3.1.

As one might guess by checking the computations for the Rindler universe, the metric for the future light cone in these coordinates is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \tau^2\,\mathrm{d}\lambda^2\,. \tag{3.3}$$

This is known as the Milne universe.

3.2 FLRW Spacetime

One might notice that Eq. (3.3) is a Friedmann–Lemaître–Robertson–Walker (FLRW) metric (Hawking and Ellis 1973, Sec. 5.3; Wald 1984, Chap. 5). Hence, it is spatially homogeneous and isotropic. A way of noticing it is by realizing that the components of Eq. (3.3) do not depend on λ, for $\left(\frac{\partial}{\partial\lambda}\right)^a$ is a Killing field (in fact, it is just the boost Killing field, which is spacelike in this region of Minkowski spacetime).

As one can read from Eq. (3.3), the scale factor of the Milne model is given by

$$a(\tau) = \tau. \tag{3.4}$$

3.3 Cosmic Observers

It is interesting then for us to ask who are the cosmic observers in the Milne universe, *i.e.*, who are the observers with four-velocities given by $u^a = \left(\frac{\partial}{\partial\tau}\right)^a$ (notice these four-velocities are appropriately normalized). To figure this out, let us begin by computing the acceleration of these observers. In order to do so, we'll need the Christoffel symbols for the Milne metric. The non-vanishing ones are given by

$$\Gamma^{\tau}{}_{\lambda\lambda} = \tau \quad \text{and} \quad \Gamma^{\lambda}{}_{\tau\lambda} = \frac{1}{\tau} \tag{3.5}$$

or can obtained from these by symmetry. This can be seen, *e.g.*, with `Mathematica` by means of a package aimed at General Relativity, such as `OGRe` (Shoshany 2021).

The acceleration of the cosmic observers will be then given by

$$a^a = u^b \nabla_b u^a, \tag{3.6a}$$
$$a^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho, \tag{3.6b}$$
$$= \partial_\tau u^\mu + \Gamma^\mu{}_{\tau\tau}, \tag{3.6c}$$
$$= 0, \tag{3.6d}$$

i.e., the cosmic observers are not accelerating. In other words, they are inertial observers.

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References with open access sources are indicated by a 🔓 next to them, which is in fact a link to somewhere in which the reference can be retrieved legally and for free.

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