Entropy Bounds and Holography in General Spacetimes

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ABSTRACT: This notebook is an introduction to entropy bounds and holography in general spacetimes. Its main purpose is to gain practice with these themes. The themes and structure are greatly inspired by the lecture series by Bousso (2018a,b,c).

KEYWORDS: quantum field theory in curved spacetime, black hole thermodynamics, general relativity.

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Contents

1 The Bekenstein Bound

We begin by discussing the early days of black hole thermodynamics. We shall only have a brief overview focusing particularly on the so-called "Bekenstein bound", but more detailed discussions can be found at the references by Almeida (2021), Belfer (2014), Wald (2019), and Wheeler (2000). From a technical perspective, our discussion is inspired by the references by Bousso (2002, 2018a,b,c, 2019), Hawking and Ellis (1973), and Wald (1984, 1994).

While many people contributed to the dawn of black hole thermodynamics, we will particularly focus on Jacob Bekenstein. His advisor, John Archibald Wheeler, telled the following anecdote (Wheeler 2000, p. 314):

> The idea that a black hole has no entropy troubled me, but I didn't see any escape from this conclusion. In a joking mood one day in my office, I remarked to Jacob Bekenstein that I always feel like a criminal when I put a cup of hot tea next to a glass of iced tea and then let the two come to a common temperature, conserving the world's energy but increasing the world's entropy. My crime, I said to Jacob, echoes down to the end of time, for there is no way to erase or undo it. But let a black hole swim by and let me drop the

> hot tea and the cold tea into it. Then is not all evidence of my crime erased forever? This remark was all Jacob needed. He took it seriously and went away to think about it.

Wheeler's gedankenexperiment is claimed to be a crime due to an earlier result known as the "no-hair theorem" (see, *e.g*, Chruściel, Costa, and Heusler 2012; Hawking and Ellis 1973, Sec. 9.3; Wald 1984, Sec. 12.3). The no-hair theorem states, roughly, that stationary black holes are completely characterized by their mass, charge, and angular momentum only. While the complete theorem is still only a conjecture, it enjoyed a few proofs in particular cases by Carter (1971, 1973), Hawking (1971, 1972), Israel (1967, 1968), and Robinson (1975), among others. Due to these uniqueness theorems, if Wheeler were to pour the hot and cold teas inside a black hole, all information about the tea apart from its mass, charge, and angular momentum would be lost. This includes the entropy contained in the tea, as it seems unlogical at first to even assume black holes could have thermodynamical properties.

Indeed, suppose a black hole was put in thermal contact with a gas, for example. Then, since the black hole cannot emit any particles, it will only absorb the particles of the gas until there is nothing left, meaning he can only be in thermal equilibrium with something at temperature $T = 0\,\mathrm{K}$. Hence, black holes would have no entropy.

In principle, one could try to accept that black holes simply have no entropy and entropy can be lost to black holes. Nevertheless, that opens the possibility that a decrease in entropy at a laboratory could be excused by claiming it was absorbed by a black hole. The second law of thermodynamics becomes observationally meaningless. Wald (2019), for example, found somewhat palatable to let go of the second law, but Wheeler found it important.

Quite recently, results by Christodoulou (1970) and Hawking (1971) showed that the area of a black hole could not decrease over time. Bekenstein (1972, 1973, 1974) then decided to use this result and the informational notion of entropy (see, *e.g*, Toffoli 2016) to boldly propose that the entropy of a black hole is actually proportional to its area. Within information theory, entropy is to be understood as "disinformation", and the black hole's area seems to be precisely a quantity measuring how much one can't see about the interior of the black hole. In symbols, Bekenstein's proposal is that

$$S_{BH} \propto \frac{A k_B c^3}{G\hbar} = \frac{A k_B}{l_P^2}, \tag{1}$$

where

$$l_P = \sqrt{\frac{G\hbar}{c^3}} \tag{2}$$

is the Planck length. Notice that Eq. (1) consists in the bold claim that a thermodynamical, stochastic quantity (the entropy) is equal to a geometric quantity (the area).

Following this, Bardeen, Carter, and Hawking (1973) and Carter (1973) proved that the second law of thermodynamics is not the only one that has a correspondence within black hole mechanics. In fact, all but the third law have geometrical proofs as analogues within black hole thermodynamics. Israel (1986) would later provide a stronger analogue for the third law. One would have the analogies shown on Table 1.

Table 1: *Analogies between the laws of thermodynamics and the laws of black hole mechanics. Units with $\hbar = c = G = k_B = 1$ are in effect.*

Law	Thermodynamics	Black Hole Mechanics
zeroth	two bodies in equilibrium have the same temperature T	a stationary black has a constant surface gravity κ on its horizon
first	$T\,\mathrm{d}S = \mathrm{d}E + \text{work terms}$	$\frac{\kappa}{8\pi}\,\delta A = \delta M - \Omega_H\,\delta J - \Phi_H\,\delta Q$
second	$\mathrm{d}S \geq 0$	$\delta A \geq 0$
third	difficult to reach $T = 0$ physically	difficult to reach $\kappa = 0$ physically

The fact that the zeroth and second laws suggest one should identify the surface gravity κ (Wald 1984, 1994, see, *e.g*,) with the temperature T and the horizon area with the entropy seems like a mere coincidence. The fact this analogy still holds when one considers the first law is more striking, especially because the black hole mass M and the energy E *are* the same physical object. Furthermore, since Ω_H represents the angular velocity of the horizon, J the angular momentum, Φ_H the electric potential at the horizon, and Q the total charge, it holds that the first law represents the work terms exactly as one would have in ordinary thermodynamics.

These similarities were at first met with skepticism and discarded as simply a curious analogy. After all, if black holes were to have entropy they should also have a non-vanishing temperature. Things changed when Hawking (1974, 1975) reluctantly showed that black holes in fact do have a temperature as a consequence of quantum effects, with the (Hawking) temperature being given by

$$T_H = \frac{\hbar\kappa}{2\pi k_B c}, \tag{3}$$

meaning the surface gravity should indeed be identified with the temperature in a physical manner. Furthermore, by means of the first law of black hole mechanics, this fixes the value of the proportionality constant in Eq. (1) on the preceding page as

$$S_{BH} = \frac{A k_B c^3}{4G\hbar}, \tag{4}$$

justifying the notation S_{BH} for the Bekenstein–Hawking entropy.

These results make the notion of black holes as thermodynamical objects being taken more seriously. Bekenstein (1972, 1973, 1974) had proposed the generalized second law (GSL), which stated that the second law of thermodynamics should be "patched" to

$$\mathrm{d}S_{\text{tot}} = \frac{k_B c^3}{4G\hbar}\,\mathrm{d}A + \mathrm{d}S_{out} \geq 0, \tag{5}$$

where A is the area of all black holes and S_{out} is the usual entropy of all matter outside black holes. Notice how this formula adds a geometrical quantity—the area—to an informational quantity—the outside entropy.

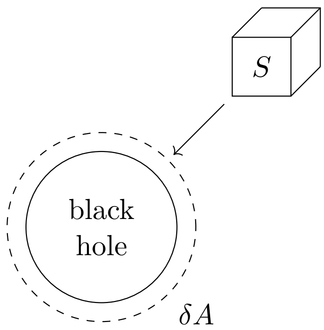


Figure 1: *Suppose some matter with entropy S falls in the black hole. For the GSL to hold, it is necessary for the growth δA of the black hole's area to be enough to compensate for the matter entropy S.*

The outside entropy can be written in terms of the von Neumann entropy (see, *e.g*, Weinberg 2015) for the matter lying outside the black hole. Let the global density matrix of all matter, inside and outside black holes, be ρ_{global}. Then we can trace over the degrees of freedom inside black holes to obtain the density matrix of degrees of freedom outside the black holes ρ as

$$\rho = \operatorname{Tr}_{\text{inside}} \rho_{\text{global}}. \tag{6}$$

We then have S_{out} being given by

$$S_{\text{out}} = -\operatorname{Tr} \rho \log \rho, \tag{7}$$

which is a measurement of disinformation (see, *e.g*, Toffoli 2016; Weinberg 2015).

The main proposal given by Bekenstein was the GSL. While recently there have been a number of attempts to give a general proof of the GSL (see, *e.g*, Wall 2009), one can still wonder why exactly it is necessary for this conjecture to hold. Consider the situation illustrated in Fig. 1, where some matter with entropy S falls into a black hole. For the GSL to hold, the area of the black hole must increase by an amount δA large enough to balance the loss of entropy S.

What is curious, though, is that the black hole is ruled by general relativity (GR) by means of the Einstein field equations (EFEs). In order to consider the quantum effects, we shall use the semiclassical Einstein field equations, reviewed in the books by Hu and Verdaguer (2020) and Wald (1994). This semiclassical version of the EFEs is written as

$$G_{ab} = 8\pi \left\langle \hat{T}_{ab} \right\rangle_{\omega}, \tag{8}$$

where $\left\langle \hat{T}_{ab} \right\rangle_{\omega}$ is the expectation value of the stress-energy-momentum tensor operator in the quantum state ω.

Let us turn the question around and ask what does the GSL implies. By forcing the process in Fig. 1 to respect the GSL we get an upper bound on the entropy content of the matter system. Namely, the GSL demands that

$$S \leq \frac{k_B \, \delta A}{4 l_P^2}. \tag{9}$$

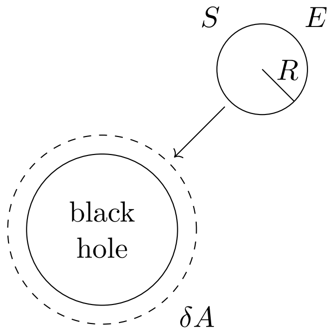


Figure 2: *A sphere of radius R, energy E, and entropy S is lowered from infinity into a black hole.*

This is curious, since the black hole cares only about the mass, charge and angular momentum of the matter block, but not at all about its entropy. Let us try to understand this in greater detail.

Suppose we have a matter sphere with radius R and energy $E \ll M$, where M is the black hole's mass. We will lower this matter sphere from infinity into the black hole, as illustrated in Fig. 2. We lower the sphere slowly with the intention of extracting work, so that we add as little mass as possible to the black hole. This is known as a Geroch process (Geroch 1971).

There is a redshift factor between the energy at infinity and the energy near the black hole. Namely, the energy at a radius r in spherical coordinates is given by

$$E(r) = E\sqrt{1 - \frac{2M}{r}} \tag{10}$$

for a Schwarzschild black hole. While one might expect to be able to lower this energy to zero by letting the sphere nearly touch the event horizon, it is important to recall that the sphere has a non-vanishing radius. Therefore, the minimum energy that can be added to the black hole is

$$E(2M + R_c) = E\sqrt{1 - \frac{2M}{2M + R_c}} > 0, \tag{11}$$

where R_c represents the radial coordinate at which the sphere is located (notice this is not the physical proper distance). Therefore, we see that it is impossible to add entropy to the black hole without adding at least some amount of energy.

While one could hope to avoid this by making the system smaller, it is important to remember that quantum mechanics (QM) eventually kicks in and leads to large energy uncertainties for small systems.

To find out exactly how much energy is being added to the black hole in terms of R, not R_c, let us recall that the geodesic radial distance (the proper distance for the sphere with $R \ll 2M$) is given by

$$\mathrm{d}l^2 = \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2\,, \tag{12a}$$

$$l(r) = \int_{2M}^{r} \frac{\mathrm{d}r'}{\sqrt{1-\frac{2M}{r'}}}, \tag{12b}$$

$$= \int_{2M}^{r} \frac{\sqrt{2M}}{\sqrt{r'-2M}} + \mathcal{O}\Big(\sqrt{r'-2M}\Big)\,\mathrm{d}r'\,, \tag{12c}$$

$$= 2\sqrt{2M(r-2M)} + \mathcal{O}\Big((r-2M)^{\frac{3}{2}}\Big). \tag{12d}$$

Therefore,

$$R_c \approx \frac{R}{8M}. \tag{13}$$

Hence, the minimum energy added to the black hole is

$$\delta M_{\min} = E(2M + R_c), \tag{14a}$$

$$= E\sqrt{1 - \frac{2M}{2M+R_c}}, \tag{14b}$$

$$= E\sqrt{\frac{R_c}{2M}} + \mathcal{O}\left(\left(\frac{R_c}{2M}\right)^{\frac{3}{2}}\right), \tag{14c}$$

$$\approx E\sqrt{\frac{R^2}{16M^2}}, \tag{14d}$$

$$= \frac{ER}{4M}. \tag{14e}$$

Since the area of a Schwarzschild black hole is $A = 16\pi M^2$, we find that

$$\delta A_{\min} = 8\pi ER. \tag{15}$$

Let us restore the relevant constants so that we read this expression better. We have

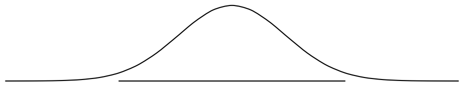
$$\delta A_{\min} = 8\pi \frac{E}{E_P}\frac{R}{l_P} l_P^2, \tag{16a}$$

$$= 8\pi E\sqrt{\frac{G}{\hbar c^5}} R\sqrt{\frac{G\hbar}{c^3}}, \tag{16b}$$

$$= \frac{8\pi GER}{c^4}. \tag{16c}$$

If we convert this area to the black hole entropy $S_{BH} = \frac{k_B A c^3}{4G\hbar}$ we get

$$\delta S_{BH} = \frac{2\pi k_B ER}{\hbar c}. \tag{17}$$



$2R$

Figure 3: *A single photon wavepacket.*

If we now impose the GSL, we find that we must have $S \leq \delta S_{BH}$. We then get to the Bekenstein (1981) bound,

$$S \leq \frac{2\pi k_B E R}{\hbar c}. \tag{18}$$

A few remarks are particularly important:

i. the black hole parameters or G do not enter the Bekenstein bound at all;

ii. while we did the argument for a sphere, it also holds for any object that fits inside the sphere, meaning the bound can be applied to any object with R being the radius of the smallest sphere in which the object fits;

iii. we had to assume the body to be weakly gravitating so that $R \ll 2M$ and $E \ll M$.

1.1 A Few Critiques on the Bekenstein bound

Let us check how exactly the Bekenstein bound holds against a couple of simple cases.

Firstly, let us consider a very simple system: a black hole. In this case, $E = M$, $R = 2M$, and $S = \frac{A}{4} = 4\pi M^2$. Hence, we have

$$S = 4\pi M^2 = 2\pi(2M^2) = 2\pi E R. \tag{19}$$

Therefore, a black hole exhausts the bound and has the maximum allowed entropy according to the Bekenstein bound.

Next consider a sphere of radiation at temperature T. Consider the sphere has radius R. Using the formulae from the book by Pathria and Beale (2022, Sec. 7.3), we can see that the energy and the entropy are given by ($c = k_B = \hbar = 1$)

$$E \propto R^3 T^4 \quad \text{and} \quad S \propto R^3 T^3, \tag{20}$$

where we are ignoring numerical constants. Notice then that $ER \propto S^{\frac{4}{3}}$. In the thermodynamics limit, $S \gg 1$, and therefore we get $S \ll ER$, as expected from the Bekenstein bound.

Nevertheless, consider now the situation in which $S \sim \mathcal{O}(1)$. For example, we can consider a single photon wavepacket such as the one illustrated in Fig. 3. Since it has two polarizations, its entropy must be about ($k_B = 1$) $S \sim \log 2$. As for the energy, Heisenberg's uncertainty principle yields ($\hbar = c = 1$) $E \sim \frac{1}{R}$.

With these estimates in mind, we notice that

$$ER \sim \mathcal{O}(1) \sim S, \tag{21}$$

and hence the details we are ignoring might be relevant to ensure the bound is violated or not. In any case, it is a tight call.

What is truly difficult about this case, and others, is to define what we actually mean by the quantities S, E, and R. There is no single objective way of defining each of these quantities simultaneously. While it is remarkable that quantum gravity considerations bound the behavior of nongravitational, low energy physics, it is frustrating that it is difficult to test it due to the challenges in even defining what the Bekenstein bound means.

An a second example, consider a hydrogen atom. We have two immediate approaches:

i. we understand the hydrogen atom as a state in the standard model, in which case we manage to get well-defined values for S and for E. Nevertheless, since the state is a state over the entire spacetime, the meaning of R becomes obscure.

ii. The alternative is to start with a well-defined R by understanding the atom as having a given size, but a localized system in quantum field theory (QFT) leads to particle creation, hence making the energy and entropy ill-defined.

There are many subtleties in the derivation of the Bekenstein bound and some arguments can be raised against it. Unruh and Wald (1982, 1983), for example, argue that the Bekenstein bound is not necessary for the GSL to hold, as a buoyancy effect occurs close to the black hole due to the radiation felt by a slowly lowered box of matter. For further details on this and other critiques, see, for example, the reviews by Bousso (2002, 2019).

Furthermore, there are some apparent counterexamples for the Bekenstein bound. For example, one can consider the Casimir effect (Casimir 1948), in which a negative energy density forms in between two conducting plates. Since we have $E < 0$ and the von Neumman entropy is always positive, it follows that the bound cannot be attained. It has been shown by Costa and Matsas (2022) though that, once one considers the struts necessary to keep the plates apart, the net energy of the Casimir system must be positive.

1.2 Spherical Entropy Bound

A second entropy bound can be derived by considering not the Geroch process, but the Sussking process (Susskind 1995). This time, instead of lowering matter into a black hole, we shall use the matter to create a black hole.

We begin with some matter of mass E and entropy S_{matter} in some spacetime. We assume that this spacetime allows the formation of black holes (for example, pick an asymptotically flat spacetime). We assume the spacetime to be sufficiently well-behaved so that one can appropriately describe a sphere around the matter. This is the case for weakly gravitating systems and for spherically symmetric spacetimes. Let A be the area of the smallest sphere that can encircle the matter and suppose the matter is stable in timescales much larger than $A^{\frac{1}{2}}$, hence allowing us to neglect any time dependence of A.

Suppose M is the mass of a black hole with area A. Then certainly $E < M$, for otherwise the system would not be gravitationaly stable. In order to collapse the matter system to a black hole, we can send it a shell of mass $M - E$. We assume this shell to be

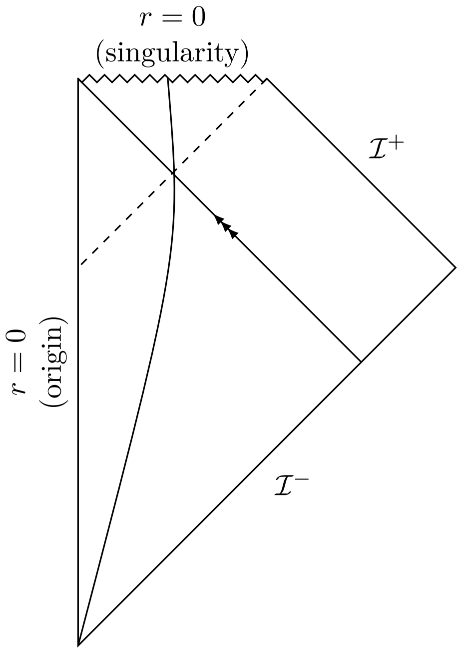


Figure 4: *Penrose diagram depicting a null spherical shell of matter with mass $M - E$ collapsing to meet the matter with mass E at the center. The event horizong formed is shown with a dashed line.*

extremely well-separated from the starting system. For example, it could be a null shell incoming from the past null infinity, similar to the Vaidya metric (Vaidya 1951). This is illustrated by the Penrose diagram in Fig. 4.

With this protocol, the initial entropy is given by

$$S_{\text{initial}} = S_{\text{matter}} + S_{\text{shell}}, \tag{22}$$

while the final entropy is

$$S_{\text{final}} = S_{BH} = \frac{A}{4}, \tag{23}$$

for there is only a black hole left.

The GSL demands that $S_{\text{final}} \geq S_{\text{initial}}$. Since S_{shell} is certainly positive, we conclude that

$$S_{\text{matter}} \leq \frac{A}{4}. \tag{24}$$

One might be worried about the extensivity of entropy leading to issues as we increase the sphere's size. However, notice that while it is true that $S \sim R^3$ and $A \sim R^2$ (where R is the sphere's radius), it is also true that the sphere of matter must satisfy $M \lesssim \frac{R}{G}$ in order to maintain gravitational stability. At constant density ρ, $M \sim \rho R^3$, and one gets a maximum size: a black hole is created before the bound is violated.

In situations where both the spherical entropy bound and the Bekenstein bound are assumed to hold, the spherical entropy bound is weaker. Indeed, for some body to possess gravitational stability we must assume $2E \leq R$ (in four dimensions). Hence

$$
\begin{aligned}
S &\leq 2\pi E R, && (25a)\\
&\leq \pi R^2, && (25b)\\
&= \frac{A}{4}. && (25c)
\end{aligned}
$$

Notice also that the Casimir effect no longer seems to present such an obvious contradiction, even without considering the energy due to the struts that hold the plates apart.

Nevertheless, there are still issues. For example, the so-called species problem. Consider once again Eq. (20) on page 7, but this time consider the field that composes the radiation is one among N possible species. In this case, we would get

$$E \propto N R^3 T^4 \quad \text{and} \quad S \propto N R^3 T^3, \tag{26}$$

which means

$$S \sim N^{\frac{1}{4}} R^{\frac{3}{4}} E^{\frac{3}{4}}, \tag{27}$$

and gravitational stability leads to

$$S \lesssim N^{\frac{1}{4}} A^{\frac{3}{4}}. \tag{28}$$

From our present knowledge, it is generous to estimate $N \sim \mathcal{O}(10^3)$. For a thermal system, we can estimate $A \gg 1$ and get $N^{\frac{1}{4}} A^{\frac{3}{4}} \ll A$ for systems much larger than the Planck scale (near the Planck scale, this description is essentially meaningless and there is no point in pursuing it so deeply).

In spite of this, as Sorkin, Wald, and Jiu (1981) and Unruh and Wald (1982), it is simple to write a QFT Lagrangian with as many species as desired. For example, there is no difficulty, in principle, for one to write a theory with 10^{50} different species of the photon. This makes it possible to violate Eq. (28) with ease. However, as criticized by Bousso (2002), the fact that this is allowed in the mathematical model does not imply it happens in nature.

1.3 Relative Entropy

Casini (2008)—building on work by Marolf, Minic, and Ross (2004), Marolf and Sorkin (2002, 2004), Sorkin (1997), Unruh (1976), and Wald (1999, 2001) among others—argued how to solve these difficulties by defining localized, but still well-defined, notions of entropy and energy. We follow Casini (2008).

Firstly, we notice that the Bekenstein bound makes no reference to gravitational phenomena. Eq. (18) on page 7 does not involve any gravitational parameters and can be interpreted as a statement concerning non-gravitational matter. Hence, it seems reasonable to pursuit a proof in flat spacetime through QFT alone.

Firstly, let us give a sensible meaning to the entropy of a localized system. We start with a quantum state ρ (assumed, for simplicity, to be a density matrix) and restrict it to a finite region of interest, V. This is done by means of a partial trace. We get to

$$\rho_V = \operatorname{Tr}_{V^c} \rho. \tag{29}$$

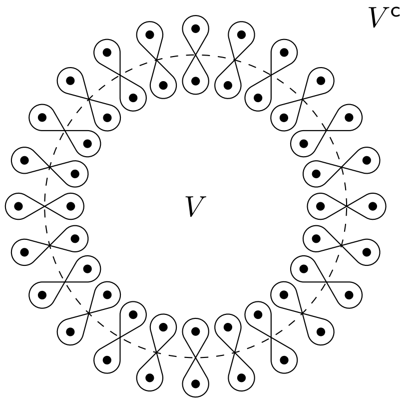


Figure 5: *Given some bounded region in space, the entropy inside it is going to diverge because there is entanglement throughout the whole boundary (dashed line).*

The entropy of ρ_V is the Von Neumann entropy, given by

$$S(\rho_V) = -\operatorname{Tr}[\rho_V \log \rho_V]. \tag{30}$$

This expression turns out to be divergent (Bombelli et al. 1986). Why is it that the entropy diverge? Because once we look at a finite region V, there is going to be entangled pairs throughout the whole boundary, as pictured in Fig. 5.

Nevertheless, it should be noted that this divergence is not due to any matter that sits inside V. Rather, the divergence is already there when the state is the vacuum state. The divergence is thus not due to the matter, but due to the state's ultraviolet behavior.

Since our interest regards only the matter entropy, we can define our entropy of interest as

$$S_V = S(\rho_V) - S(\rho_V^0), \tag{31}$$

where ρ_V^0 is the vacuum state restricted to the region we are interested in. As long as ρ is a Hadamard state, S_V will be a localized and finite notion of entropy. While one needs some sort of regularization to make sense of such an expression (such as putting the theory on a lattice), the resulting S_V will be independent of the regularization scheme. Hence, this seems like an adequate description of what the entropy going into the Bekenstein bound should be.

Next we move on to understand how to prescribe a good notion of localized energy. However, there is a further difficulty. Let write the Bekenstein bound as

$$S \leq \lambda E R. \tag{32}$$

While we want now to make sense of E, there is also a disagreement on what is the value of λ—2π according to the original derivation by Bekenstein (1981), but somewhat larger according to later arguments also given by Bekenstein (1994)—or what is the meaning of R—the radius of a circumscribing sphere according to Bekenstein (1981) or the shorter dimension according to Bousso (2003). Hence, Casini (2008) instead decides to prescribe a well-defined candidate for the product $\lambda E R$ as a whole.

Consider once again the vacuum density matrix restricted to V. We define an operator K according to

$$\rho_V^0 = \frac{e^{-K}}{\mathrm{Tr}[e^{-K}]}. \tag{33}$$

Within algebraic quantum field theory, K is known as a modular Hamiltonian (see, for example, Correa da Silva 2018; Haag 1996).

In the case of a quantum field in the Rindler spacetime (Rindler 1966), the form of the modular Hamiltonian is known exactly. It is given by (Bisognano and Wichmann 1976; Unruh 1976)

$$K = 2\pi \iiint_{z \geq 0} \mathcal{H}(0, x, y, z) z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,, \tag{34}$$

where V is the $z \geq 0$ region of Minkowski spacetime and $\mathcal{H}(t, x, y, z)$ is the Hamiltonian density operator. Furthermore, this holds for any QFT, for it follows from the Wichmann axioms (Bisognano and Wichmann 1975, 1976; for a review, see Haag 1996).

Suppose now the amount of matter we are interested in is located far away from the boundary at $z = 0$. In this case, Eq. (34) reduces to approximately

$$K \sim 2\pi E R, \tag{35}$$

which is exactly the expression we are after. In more complicated scenarios K will likely be more complicated, and might not even be local.

There are two reasons for us to not immediately consider K as the right-hand side of the Bekenstein bound.

- it is an operator, not a number, so we should at least take its expectation value;
- it is ill-defined by an additive constant, as one can notice from Eq. (33).

Hence, what we will consider for the right-hand side of the Bekenstein bound is the difference

$$K_V = \mathrm{Tr}[K\rho_V] - \mathrm{Tr}\Big[K\rho_V^0\Big], \tag{36}$$

where we are subtracting the expectation value of K in the vacuum so we can get rid of the additive constant ambiguity.

Therefore, the Bekenstein bound proposed by Casini (2008) is now

$$S(\rho_V) - S(\rho_V^0) \leq \mathrm{Tr}[K\rho_V] - \mathrm{Tr}\Big[K\rho_V^0\Big]. \tag{37}$$

Recalling the expression for the Von Neumann entropy, that $K = -\log \rho_V^0 - \log\Big(\mathrm{Tr}\Big[e^{-K}\Big]\Big)$, and that $\mathrm{Tr}[\rho_V^0] = \mathrm{Tr}[\rho_V] = 1$, we find that Eq. (37) is equivalent to

$$\mathrm{Tr}[\rho_V \log \rho_V] - \mathrm{Tr}\Big[\rho_V \log \rho_V^0\Big] \geq 0. \tag{38}$$

Now, the left-hand side of Eq. (38) is known in quantum information as the relative entropy between ρ_V and ρ_V^0, and it is written as (Nielsen and Chuang 2011, Eq. (11.50))

$$S(\rho_V \| \rho_V^0) = \mathrm{Tr}[\rho_V \log \rho_V] - \mathrm{Tr}\Big[\rho_V \log \rho_V^0\Big]. \tag{39}$$

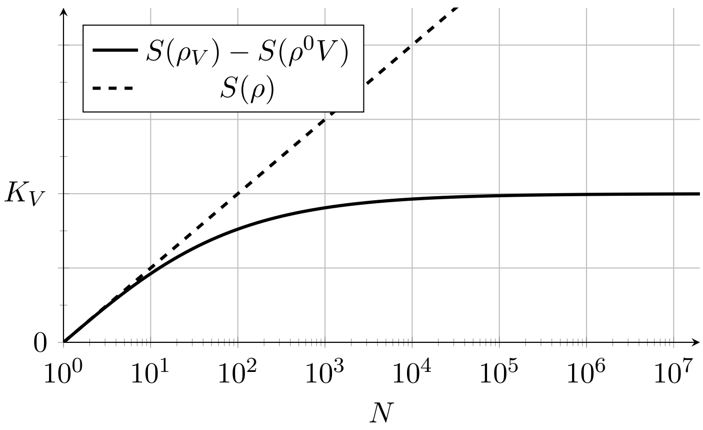


Figure 6: *While the global entropy $S(\rho)$ grows logarithmically with the number of species, the vacuum subtracted entropy within a region V is bounded from above.*

For finite quantum systems (and we can assume we are dealing with a limiting case of a finite system by putting the theory on a lattice), it holds that (Nielsen and Chuang 2011, Theorem 11.7)

$$S(\rho_V \| \rho_V^0) \geq 0, \tag{40}$$

with $S(\rho_V \| \rho_V^0) = 0$ if, and only if, $\rho_V = \rho_V^0$.

Therefore, within the framework given by Casini (2008), the Bekenstein bound is a mere consequence of the relative entropy being positive-definite.

The species problem is thus solved within this analysis. Indeed, while adding more species increases the entropy $S(\rho_V)$, it also increases the entropy $S(\rho_V^0)$. Even if their different keeps growing with the number of species N, the very fact that the relative entropy must be positive bounds the difference growth from above. It is correct that the global entropy $S(\rho)$ will keep growing logarithmically with N, but the vacuum subtracted entropy within V, S_V, never gets out of control. This is pictured in Fig. 6.

We also get an alternative solution to the Casimir energy puzzle. Namely, while the Von Neumann entropy is necessarily positive, S_V does not have to be. One can lose entropy, for example, by annihilating Rindler modes from the vacuum.

While this approach definitely takes a step forward in understanding entropy bounds, Casini (2008) used an analysis that seems quite focused in non-gravitational systems. What can be done in a general spacetime?

2 The Bousso Bound

In order to consider gravitational effects directly, let us go back to an entropy bound that actually depends on G: the spherical entropy bound. It essentially states that the boundary area of some region determines the number of degrees of freedom within the region, hence allowing us to picture "the world as a hologram" ('t Hooft 1994; Susskind 1995).

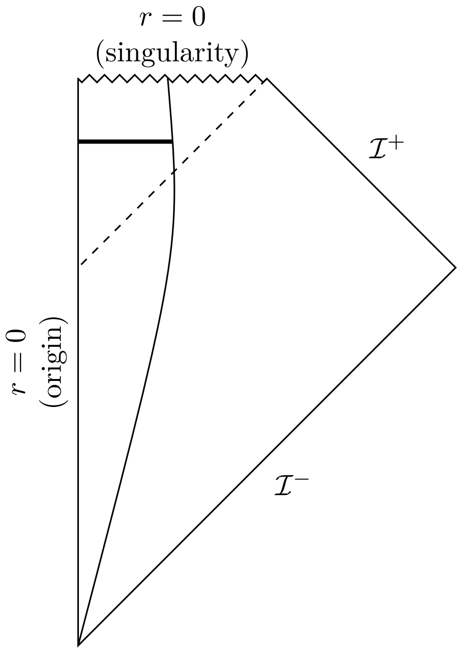


Figure 7: *The ball of matter represented by the thick line inside the black hole cannot collapse to a black hole of the same size. Hence, the spherical entropy bound is violated for late times.*

While Casini's approach managed to solve issues regarding the species problem, there are issues that occur in curved spacetime that we still have not pointed out and that lead to other issues. Let us give a few examples.

Firstly, we had previously argued that scaling a ball of matter would lead to a black hole before violating the spherical entropy bound. This can fail, because is is not always possible to to create a black hole of with the same area of the matter we are interested in. For example, consider the diagram in Fig. 7, which depicts a ball of matter inside an already formed black hole. For example, this can be the case of a star that collapsed to a black hole. In this situation, the star within the black hole cannot collapse to a black hole with the same area. As a consequence, the spherical entropy bound can be violated at late times.

Next, consider a closed universe. Pick a matter system surrounded by some spherical surface, as depited in Fig. 8 on the following page. The same area that bounds the "interior matter" also bounds the "exterior matter", but it seems unlikely that, say, the surface of the Earth is enough to bound all of the entropic content in the rest of the Universe.

A third counterexample was propose by Fischler and Susskind (1998), and consists simply of picturing some spherical region in a flat, expanding universe. The region is subjected to expansion at a fixed entropy density and, as a consequence, eventually the bound is violated.

Last, one could also measure the area in an accelerated frame of reference, which

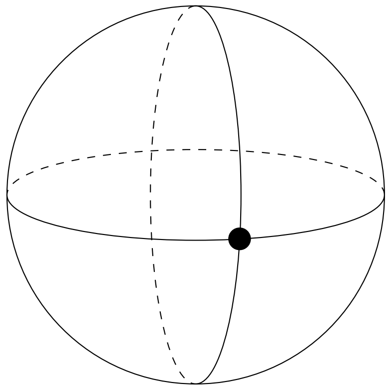


Figure 8: *In a closed Universe, it seems illogical for the small area around the black, "inner" region to bound the gargantuan entropy of the white, "outer" region.*

would diminish the area due to Lorentz contraction, possibly leading to a violation*.

Bousso (1999a)—building on the works of Bekenstein (1981), Corley and Jacobson (1996), Fischler and Susskind (1998), 't Hooft (1994), and Susskind (1995) among others—proposed a "covariant entropy conjecture", nowadays known as the Bousso bound. We shall review it now by following Bousso (1999a, 2002).

The first step to attempt at finding an entropy bound is to choose whether we shall follow Bekenstein's formula, which uses energy to bound the entropy, or the spherical entropy bound formula, which bounds the entropy with the area of surface enclosing it. Since GR does not have a local notion of energy and global notions require specific asymptotic structures—such as asymptotic flatness. We want to find a bound that works in any spacetime, which then requires us to bound the entropy by an area. The area of any two-dimensional surface has an objective definition in terms of the induced metric on the surface.

Once we have chosen the "class" of entropy bound we are aiming at, our next goal is to give meaning to the expressions it involves. We expect to find that, in some sense,

$$S \leq \frac{A}{4}. \tag{41}$$

The issue is not with the equation itself, but rather with the meaning of the quantities it involves. We must give meaning to A by choosing which two-dimensional surfaces are allowed and we must give meaning to S by explaining how the entropy is to be measured.

We will be general and conjecture an entropy bound that should hold for any two-dimensional spacelike surface B (for boundary). This includes both closed and open surfaces. The area of B is surely well-defined, for one can compute it with the metric induced on B by the spacetime metric.

The entropy should somehow relate to B. Hence, the candidates for where the entropy should be measured are the hypersurfaces H in which B is imbedded. It is meaningless

*Of course, it is certainly not trivial to discuss how thermodynamics quantities behave under changes of reference frames (Landsberg and Matsas 1996).

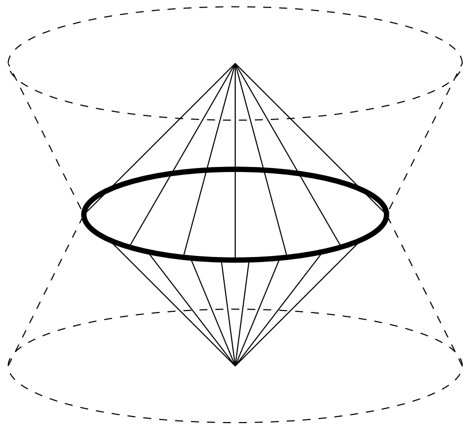


Figure 9: *Given a surface B (thick ring), there are at most four congruences of null geodesics that are orthogonal to B. The solid lines represent incoming null geodesics, while the dashed lines represent outgoing null geodesics. Time runs upwards in this diagram and one spatial dimension is suppressed.*

to try a timelike hypersurface: have a particle following a worldline on the hypersurface and there would be infinite entropy contained in it. Instead, we want to have a notion of entropy at a single instant. This leaves us with either spacelike hypersurfaces or null hypersurfaces. The previous conterexamples all involved the use of spacelike hypersurfaces, and therefore we shall impose that H is null.

Therefore, we will work with null hypersurfaces H bounded by B. As one can notice from the causal structure nearby a surface B, there are at most* four congruences of geodesics that are orthogonal to B, illustrated on Fig. 9. It seems natural to choose these hypersurfaces as being preferred.

In order to avoid problems such as the one found in the closed universe, we must also be able to select what is inside B and what is outside B. Hence, we want to find a geometric way of determining whether a given hypersurface points inwards or outwards from B. There is, though, some initial difficulty in defining "inside" and "outside" for a general surface, especially in situations such as that of a closed universe.

To bypass this, Bousso (1999a, Sec. 2) proposes a notion that is intuitive in Euclidean geometry. Start with a closed surface B. Move the surface infinitesimally along the geodesics orthogonal to B toward one of the sides of B. If the motion was inward, then the area of the surface must decrease. If it was outward, then the area must increase.

This intuition has a natural generalization within GR. Namely, we shall define "inside" and "outside" by using the notion of expansion of a congruence of geodesics (Hawking and Ellis 1973, Sec. 4.2; Wald 1984, Sec. 9.2), denoted θ. We are thus interested in hypersurfaces generated by a congruence of geodesics with $\theta \leq 0$, and should stop following them as soon as $\theta > 0$. This motivates the following definition.

*If B is chosen to be the Universe's spatial boundary, there are fewer.

Definition 1 [Lightsheet]**:**

Let (M, g_{ab}) be a four-dimensional spacetime satisfying the EFEs and let $B \subset M$ be a two-dimensional spacelike surface. A null hypersurface H is said to be a *lightsheet for B* if, and only if, it is bounded by B, its generators are orthogonal to B and their expansion is everywhere non-positive when measured in the direction away from B. ♠

We still need to discuss a few more conditions to state the conjecture. Namely, we should discuss which spacetimes are to be allowed. GR is fairly generous with the geometries it allow, and not all of them can be considered physical. Bousso (1999a) then imposed two conditions:

i. the dominant energy condition (DEC) holds (see, *e.g*, Hawking and Ellis 1973, Sec. 4.3; Wald 1984, Sec. 9.2);

ii. the spacetime does not possess naked singularities (see, *e.g*, Choquet-Bruhat 2009, Sec. XIII.7.1).

The first requirement is chosen in order to avoid, for example, a superluminal entropy flow. The second requirement is meant to protect us from entropy leaving the universe through these unlocked doors.

Bringing all of this together, we arrive at the following conjecture:

Bousso's Entropy Bound Conjecture:

Let (M, g_{ab}) be a four-dimensional inextendible spacetime satisfying the EFEss, the DEC, and non-nakedly singular (for a rigorous definition of nakedly singular, see, for example, Choquet-Bruhat 2009, Sec. XIII.7.1). Let $B \subset M$ be a connected two-dimensional surface and denote its area by A. Let H be a lightsheet for B. Then the total entropy S contained on H is such that

$$S \leq \frac{A}{4} \tag{42}$$

holds. □

Bousso (1999a) originally referred to this conjecture as the "covariant entropy conjecture" and later as the "covariant entropy bound" (Bousso 2018b). Other works also refer to the same conjecture as the "Bousso bound", and we shall adopt this nomenclature.

2.1 Examples

Let us now see how the Bousso bound holds against the counterexamples we had for the spherical entropy bound.

The first test refers to the case of a collapsing star. If we pick the same diagram of Fig. 7 on page 14 and draw the lightsheets for the surface that bounds the sphere at some time instant we get the diagram in Fig. 10 on the next page. This time, the lightsheets both point to the future, because they are trapped surfaces. One of them does not intersect the star at all, while the other one only intersects a portion of it.

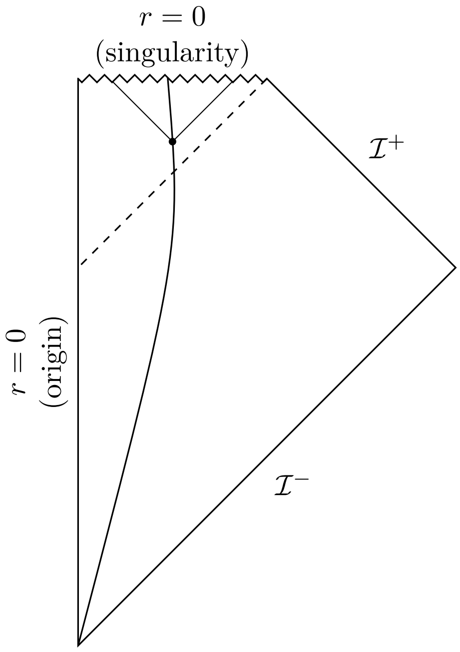


Figure 10: *Pick a spherical shell surrounding the collapsing star inside the black hole (represented by the solid circle). The only two lightsheets point to the future and are trapped surfaces (solid lines emanating from the sphere): one of them does not intersect the star at all while the other one intersects only a portion of it.*

For the closed universe, the condition that $\theta \leq 0$ picks the correct side of the boundary. It is no longer allowed for one to try to use a small area to bound the entropy of the majority of the Universe.

For a flat universe, we can draw the Penrose diagram in Fig. 11 on the following page. This time, we argument we previously gave fails because the lightsheets do not intersect all of the matter "inside" the sphere. As the sphere grows with the universe's expansion, eventually it crosses the apparent horizon (Hawking and Ellis 1973, Sec. 9.2; Wald 1984, Sec. 12.2) and the lightsheets start pointing to the same spatial direction, rather than to the same time direction (the past). Therefore, the bound is sustained because the matter content crossing the lightsheets is small.

Finally, since the Bousso bound is built entirely on geometrical considerations, it cannot be violated by a simple change of reference frames.

While we have discussed a qualitative analysis, some quantitative calculations can be found in the original paper by Bousso (1999a).

2.2 Generalized Bousso Bound

Is it also interesting to point out that the Bousso bound can be refined to a stronger statement.

Start with a boundary surface B. Then consider a lightsheet H. However, split H in

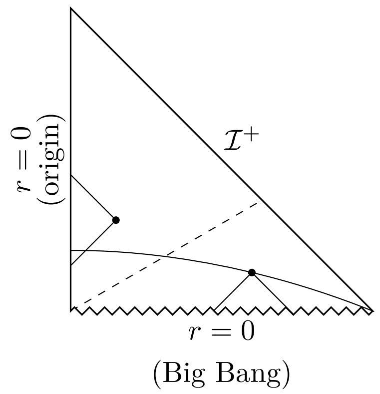


Figure 11: *Pick a spherical shell within a spacelike Cauchy surface in a flat universe. The only two lightsheets point to the past and are anti-trapped surfaces (solid lines emanating from the sphere): regardless of which of them we pick, not all matter in the inside region (as viewed from the Cauchy surface) intersects the lightsheets, and hence the entropy can indeed be bounded even in an expanding flat universe. The apparent horizon (dashed line) sets the boundary at which the surfaces are no longer antitrapped due to being too close to the origin, and in that region the bound holds precisely because there is little matter to cross those lightsheets. The apparent horizon is drawn for the equation of state $p = \rho$.*

two pieces, L and L'. L' includes the caustic at which H terminates, and its boundary is a different spatial surface B'. Hence, we have for B' that

$$S' \leq \frac{A'}{4}, \tag{43}$$

where A' is the area of B' and S' is the entropy in L'. Notice, though, that

$$S + S' \leq \frac{A}{4}, \tag{44}$$

where S is the entropy in L and A is the area of B. As a consequence of the previous equations,

$$S \leq \frac{A - A'}{4}, \tag{45}$$

and we see that the entropy of a lightsheet is limited by the different in areas of its boundaries. This was first pointed out by Flanagan, Marolf, and Wald (2000), who also provided proofs of the Bousso bound under some circumstances.

2.3 Holography in General Spacetimes

It is remarkable that the Bousso bound makes no reference to a time direction. Hence, it holds true to both the future and the past, without the need to invoke the GSL. Due to this, one should understand it not as putting a restriction on the thermodynamic entropy, but rather on the statistical or information entropy crossing a lightsheet. As a consequence, it can be seen as a motivation for a holographic principle in general spacetimes, rather than as a consequence of one.

More explicitly, we can conjecture the holographic principle for a general spacetime following Bousso (1999b). We begin by defining the number of degrees of freedom of a quantum system as $N_{\text{dof}} = \log \dim \mathcal{H}$, where $\mathcal{H}$ is the system's Hilbert space. This definition is motivated by the fact that N_{dof} is then ($\log 2$ times) the number of bits necessary to specify the state of the system. With this and the Bousso bound in mind, we formulate the following conjecture (Bousso 1999b).

Holographic Principle:

Consider a connected spatial two-surface B. Let A denote its area. Let H be a lightsheet for B and let $\mathcal{H}$ denote the Hilbert space describing all physics on H. Then it holds that $\dim \mathcal{H} \leq \exp\left(\frac{A}{4}\right)$. □

Notice that the holographic principle by no means implies the existence of a non-gravitational theory living in the spacetime's boundary. While it can be seen as necessary for the existence of such a duality, it is certainly not sufficient. Notice, thus, that there is a difference between saying a theory is "holographic" (and hence information about the bulk "fits" in the boundary) and saying it is "dual" (and hence the theory on the bulk can be matched to a theory on the boundary).

Our main question in this section is now to answer which collection of two-dimensional surfaces we should pick in order to access information about the whole spacetime. For concreteness, let us start with a popular case: anti-De Sitter (AdS) spacetime. Due to the famous conjecture by Maldacena (1999), there is some expectation that we should be able to somehow store the information about the bulk of the spacetime in its boundary. We roughly follow Bousso (1999b, 2002). Nevertheless, we shall use the terms "holographic screen" and "leaves" as later used by Bousso and Engelhardt (2015a,b).

We assume some previous knowledge about AdS, which can be reviewed in the book by Hawking and Ellis (1973, Sec. 5.2). Let us start by drawing the Penrose diagram for AdS spacetime. It is shown in Fig. 12a on the next page. It is interesting to notice that all surfaces in AdS spacetime are normal, meaning they are not trapped nor anti-trapped. If we choose to pick all surfaces at the boundary, we can use their lightsheets to foliate spacetime according to two different options: using the past or the future lightsheets—see Figs. 12b and 12c on the following page. The Bousso bound then lets us know that the AdS boundary can store all information about the spacetime at less than one bit per Planck area.

We shall say that the boundary of AdS spacetime is an example of a (holographic) screen hypersurface, which is a hypersurface containing information about the complete spacetime. Each screen hypersurface is comprised of a collection of holographic screens, which are two-dimensional surfaces containing all information about a particular light cone. In the AdS example, the holographic screens we chose were the surfaces at the boundary. The light cones encoded in each surface are the past (or future) light cones for points at the spacetime's origin. There are often many choices of screen hypersurfaces: while in the AdS case they coincide, notice that in principle we could choose to work with

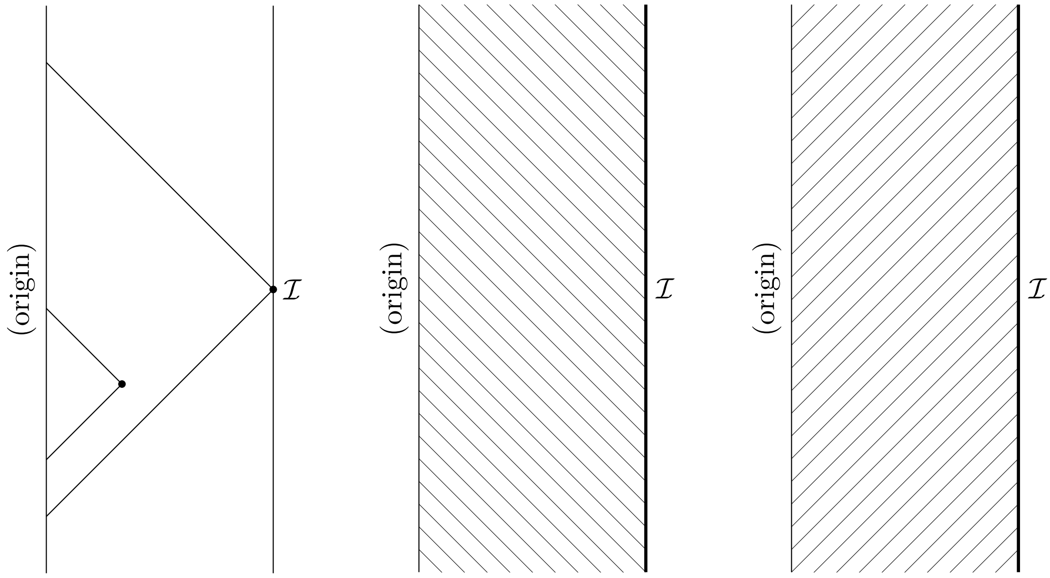


Figure 12: *Penrose diagram for AdS spacetime (see,* e.g*, Hawking and Ellis 1973, Sec. 5.2). Fig. 12a exhibits how surfaces in AdS are always normal,* i.e*, they are not trapped nor anti-trapped. Hence, the lightsheets point toward the origin. Figs. 12b and 12c illustrate that by considering all surfaces laying at the boundary we can foliate the spacetime with lightsheets in two possible ways. Notice that each null hypersurface in these foliations can be seen as a future or past light cone for a particle at the origin. In all diagrams the thick line represents a screen hypersurface.*

either past or future light cones.

Our next example is a simple case: Minkowski spacetime. Once again, all screens are normal. In order to get information about the spacetime on its boundary, we foliate it in terms of null hypersurfaces. Interestingly, Minkowski spacetime admits two different foliations that lead to two different holographic screen hypersurfaces, all shown in Fig. 13 on the next page. One of the foliations leads to a holographic screen hypersurface at $\mathcal{I}^+$ and the other at $\mathcal{I}^-$.

Next we consider a spatially closed, dust-filled Friedmann–Lemaître–Robertson–Walker (FLRW) universe. Its Penrose diagram is shown on Fig. 14 on page 23. This time there is no boundary for us to store information in, but we can still store information in one or more hypersurfaces. To do so, we pick a null foliation and, along each null hypersurface, we search for a screen in which the geodesic expansion vanishes. Such a screen is interesting because it has more than two lightsheets, as illustrated in Fig. 14a on page 23. It deserves a special name: a preferred screen. The fact that preferred screens have lighthseets in two opposing directions (future-incoming and past-outgoing, for example) means they can store information about a whole hypersurface of the null foliation. By repeating this process for each sheet in the foliation we get a locus of spheres which can encode information about a whole hypersurface for the whole foliation. Therefore, this locus is a

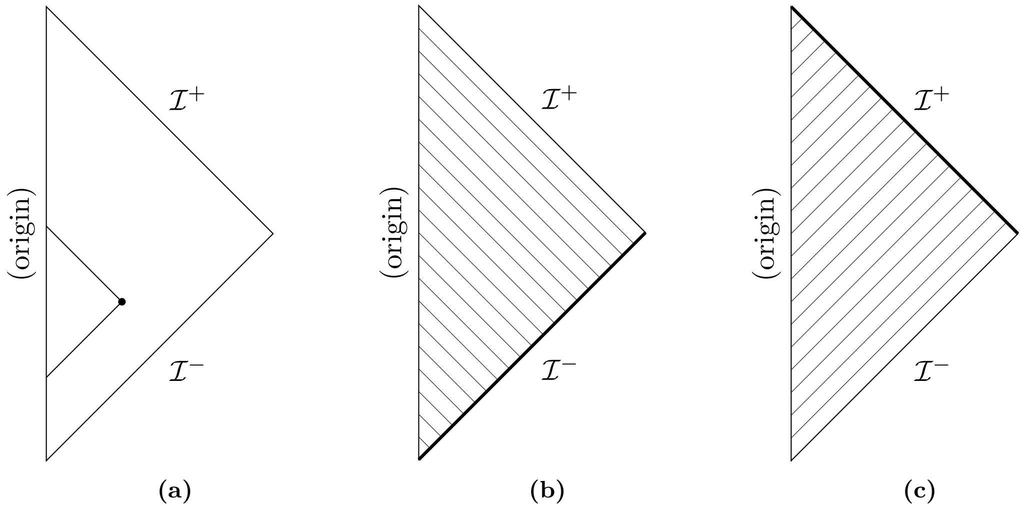


Figure 13: *Penrose diagram for Minkowski spacetime. Fig. 13a represents the lightsheets of some spherical surface within the spacetime. Figs. 13b and 13c show how one can foliate Minkowski spacetime using null hypersurfaces and how each foliation can be projected onto a different holographic screen hypersurface.*

preferred holographic screen hypersurface and encodes information concerning the whole universe. Figs. 14b and 14c on the following page illustrate two examples.

Further examples and discussion can be found in the papers by Bousso (1999b, 2002). The general recipe for building a holographic screen hypersurface for some given spacetime is the following

i. draw a Penrose diagram;

ii. choose a null foliation;

iii. identify the lightsheets for each point in the diagram;

iv. find the preferred screens at each hypersurface (they might often lie at the boundary);

v. the holographic screen hypersurface is given by the union of the individual holographic screens.

It is also interesting to point out that an improved definition of the notion of a preferred holographic screen hypersurface allows for one to derive an area theorem ensuring the area of the holographic screens increases or decreases monotonically as one follows the hypersurface. See the papers by Bousso and Engelhardt (2015a,b) for details.

3 The Quantum Focusing Conjecture

So far, most of our work has been done within the classical regime. While a few times we invoked QM to interpret results in terms of bits, or in order to motivate the GSL, we

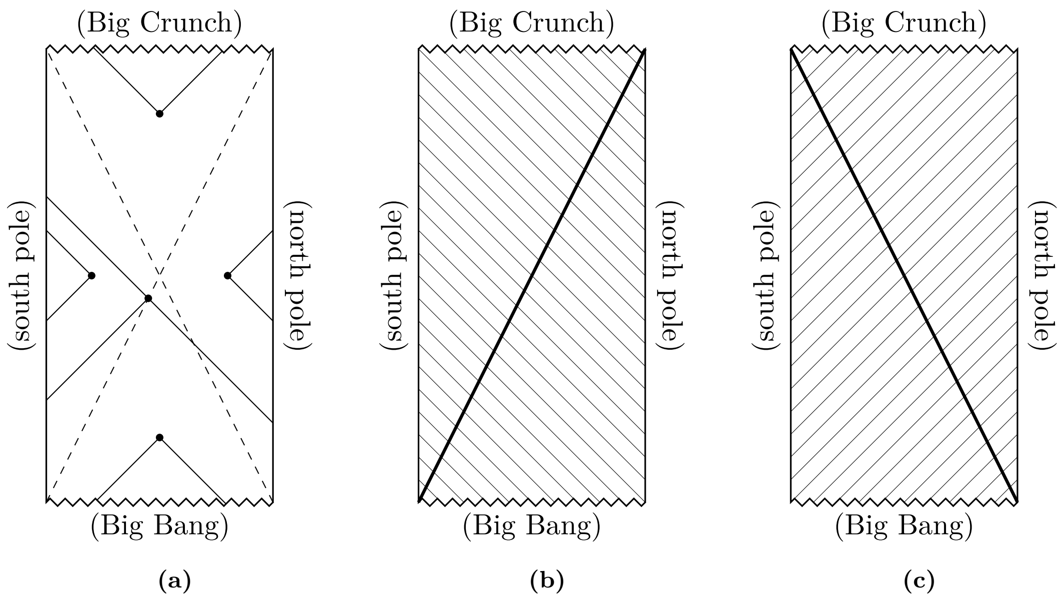


Figure 14: *Penrose diagram for a spatially closed, dust-filled FLRW universe. Fig. 14a shows how the lighsheets behave in each part of the spacetime, and in particular how there are apparent horizons on which a screen can have more lightsheets than usual. In these cases, we can store a whole sheet of a null foliation in a single screen. Figs. 14b and 14c show two ways of encoding the entire spacetime within a single holographic screen sheet.*

haven't really dived too much into quantum effects. Furthermore, all of our discussion is currently focused on entropy. We shall now try a different route. As pointed out by Bousso (2018b), we can try to invert the historical logic and think of generalized entropy as a correction on the notion of area, instead of the notion of entropy. Our discussion follows the original paper by Bousso, Fisher, Leichenauer, et al. (2016).

Historically, the ordinary second law of thermodynamics was discovered far before Hawking's area theorem. We may thus wonder how it would have been like if the area theorem came first. Eventually, we could expect someone to propose that quantum corrections to the classical, geometric notions of area take the form of the mysterious notion called entropy, and the thermodynamic second law would follow from the attempt to preserve the area theorem. This is the essence of the following steps: to understand new physics by performing quantum corrections to GR. Therefore, we define the quantum corrected area, A_Q, as

$$A_Q = \frac{4G\hbar}{k_B c^3} S_{\text{gen}} = A_{\text{BH}} + \frac{4G\hbar}{k_B c^3} S_{\text{matter}}. \tag{46}$$

It is interesting to notice that the definition given on Eq. (46) can be taken for any Cauchy-splitting surface, not only for a black hole's event horizon. The term "Cauchy-splitting" means that the surface must be such that it splits a Cauchy surface in two disjoint portions.

Furthermore, it is interesting to notice that while S_{matter} and $\frac{1}{G}$ can be divergent (due to G's renormalization group flow, for example), the combination A_Q turns out to be finite and independent of regularization scheme (see, *e.g*, the appendix of Bousso, Fisher, Leichenauer, et al. 2016, and references therein). Hence, it might hint at a glimpse of quantum gravity. Nevertheless, it should be pointed out that it still is a semiclassical concept and should be understood as such.

3.1 Classical Focusing Theorem

In order to understand areas in GR, one often employs the Raychaudhuri (1955) equation (see, *e.g*, Hawking and Ellis 1973, Sec. 4.2; Wald 1984, Sec. 9.2). It states, for null geodesics, that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{cd}k^c k^d. \tag{47}$$

In this equation, θ is the geodesic expansion, $\hat{\sigma}_{ab}$ is the shear, $\hat{\omega}_{ab}$ is the twist of the congruence of geodesics, k^a is the null vector tangent to the congruence, and λ is an affine parameter. R_{ab}, naturally, is the Ricci tensor. For hypersurface-orthogonal geodesics, which is the case we're interested in, $\hat{\omega}_{ab} = 0$.

It holds that $\hat{\sigma}_{ab}\hat{\sigma}^{ab} \geq 0$. Hence, if the null energy condition (NEC) holds, which means $R_{cd}k^c k^d \geq 0$ for any null vector k^a, then

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 \leq 0. \tag{48}$$

Hence, as long as the NEC holds, gravity is attractive in a very specific way. The exception would be only at caustics, which are points at which $\theta \to -\infty$. This is known as the classical focusing theorem.

Nevertheless, quantum effects such as the Casimir effect or the Hawking effect violate the NEC. In fact, violation of the NEC is precisely why a black hole emitting Hawking radiation gets to shrink without violating the area theorem. Therefore, it is interesting to wonder about what happens to the classical focusing theorem when the NEC no longer holds. THis will be particularly interesting in situations involving quantum mechanical systems, since the Casimir effect, for example, violated the NEC. Hence, our goal is to get to a quantized version of the classical focusing theorem.

3.2 Quantum Focusing Conjecture

In order to formulate the quantum focusing conjecture (QFC), we shall begin by attempting to replace the classical area in the classical focusing theorem with the quantum-corrected area A_Q.

The classical area enters the classical focusing theorem through the formal expression

$$\theta = \lim_{\mathcal{A}\to 0} \frac{1}{\mathcal{A}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\lambda}, \tag{49}$$

where $\mathcal{A}$ is the cross-section area of the congruence of geodesics under consideration. While θ is classically a local quantity, when we substitute the area for a quantum area we

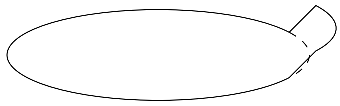


Figure 15: *By making a small functional variation to $V(y)$, we can slightly modify a region of σ. The original surface includes the dashed line, while the modified surface follows two null paths amid the spacetike portions.*

will need to consider the notion of an "outside entropy", which will then spoil the locality of the quantum expansion.

To have the notion of a quantum area, we need to first consider a Cauchy-splitting surface σ, which may or not be closed and may or not be compact. Such a surface allows us to split a Cauchy surface Σ into the two portions Σ_1 and Σ_2, referring to the two sides delimited by σ. We can choose either Σ_1 or Σ_2 to play the role of "Σ_{out}" without affecting our results. Furthermore, just as in our original discussions of the Bousso bound, we have four possible choices for a null hypersurface originating at σ and orthogonal to it. Therefore, in total, we got eight different choices on how to proceed: two choices of "outside" and four choices of null hypersurface. Bousso, Fisher, Leichenauer, et al. (2016) also states that any of these four null hypersurfaces will eventually be terminated by a caustic or whenever null generators orthogonal to σ intersect. As a consequence, these null hypersurfaces are components of the boundary of the past or future of σ.

For each point y in σ, there is a generator of the chosen null hypersurface N. We can follow this generator for an affine time λ until we get to a new point at N. Through this procedure, we can get a coordinate system at N labeled by (y, λ). We assume $\lambda = 0$ at σ and that λ grows away from σ.

Consider now the positive definite function $V(y) \geq 0$ that maps each point $y \in \sigma$ to a point in N with $\lambda = V(y)$. Any slice of N given by $V(y) = \lambda$ is Cauchy splitting. The slice $V(y) = \lambda$ is not necessarily spacelike, but might have null regions*.

We can thus build a quantum area functional such that

$$A_Q[V(y)] = A[V(y)] + 4S_{\text{out}}[V(y)]. \tag{50}$$

As we vary the functional V (for example, by scaling the parameter λ along a particular geodesic starting from σ), we can alter the value of the quantum area both by changing the shape of the two-surface and by changing what is considered "outside". Functionally, this can be done by considering the variation

$$V_\epsilon(y) = V(y) + \epsilon\vartheta(y), \tag{51}$$

where $\vartheta(y)$ is some function (for example evaluated to 1 close to some point $y_1 \in \sigma$ and vanishing elsewhere) and ϵ is some small parameter. This is illustrated in Fig. 15.

Notice then that the variation of the quantum area at $y_1 \in \sigma$ can be formally written as

$$\left.\frac{\mathrm{d}A_Q}{\mathrm{d}\epsilon}\right|_{y_1} = \lim_{\epsilon\to 0} \frac{A_Q[V_\epsilon(y)] - A_Q[V(y)]}{\epsilon}. \tag{52}$$

*We are using the definition of Cauchy surface given by Wald (1984), which allows for a Cauchy surface to have null regions, rather than being entirely spacelike as demanded by Hawking and Ellis (1973).

This would lead to the formal definition of the quantum expansion as being

$$\Theta[V(y); y_1] \equiv \lim_{\mathcal{A}\to 0} \frac{1}{\mathcal{A}} \left.\frac{\mathrm{d}A_Q}{\mathrm{d}\epsilon}\right|_{y_1}, \tag{53}$$

where $\mathcal{A}$ is the geometrical area of the support of $\vartheta(y)$.

This notion can be given a formal definition in terms of a functional derivative. We then have

$$\Theta[V(y); y_1] \equiv \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta A_Q}{\delta V(y_1)}, \tag{54}$$

where g^V_{ab} is the induced metric in σ. It occurs to ensure the functional derivative is taken with respect to geometrical area rather than with respect to coordinate area.

Notice that, as opposed to θ, Θ is non-local: it depends on the values of $V(y)$ away from y_1 and depends on the matter in Σ_{out}.

In the classical regime, S_{matter} will not contribute to the quantum area and we recover the classical expansion.

Quantum Focusing Conjecture:

Consider a generic spacetime and the conditions previously stated in order to define Θ. *We conjecture that*

$$\frac{\delta\Theta[V(y); y_1]}{\delta V(y_2)} \leq 0. \tag{55}$$

Or, in words, the quantum expansion along a direction y_1 *of some surface cannot increase when the surface is slightly deformed along the direction* y_2. □

Let us now figure out some of the implications of the QFC.

3.3 Quantum Bousso Bound

An interesting property of the QFC is the fact it implies a quantum version of the generalized Bousso bound.

We start with a two-surface σ and follow it along N until we reach a second surface σ'. We suppose σ is given by $V(y) = 0$, while σ' corresponds to some value of $V(y) \geq 0$.

Let us begin by assuming that, at $y_1 \in \sigma$, the quantum expansion is nonpositive or negative. Integrating the QFC then allows us to conclude that

$$\Theta[0; y_1] \leq 0 \Rightarrow \Theta[V(y); y_1] \leq 0, \tag{56}$$

where the equalities hold if, and only if, them both hold.

Suppose now we are interested in a second spatial surface σ' defined by $V(y)$ where

$$V(y) = \begin{cases} V(y) \geq 0, & \text{if } \Theta[0, y] \leq 0, \\ V(y) = 0, & \text{if } \Theta[0, y] > 0. \end{cases} \tag{57}$$

Due to Eq. (56) on the previous page, it follows that the quantum area increases at the slices $\alpha V(y)$, with $0 \leq \alpha \leq 1$. Hence, the quantum area of σ' is smaller than that of σ. Thus,

$$A_Q[\sigma'] \leq A_Q[\sigma]. \tag{58}$$

Notice this is a statement about quantum expansion that is only possible due to the QFC, and hence it goes beyond the original Bousso bound.

As a consequence,

$$4S_{\text{matter}}[\sigma'] + A[\sigma'] \leq 4S_{\text{matter}}[\sigma] + A[\sigma], \tag{59}$$

from which we conclude

$$S_{\text{matter}}[\sigma'] - S_{\text{matter}}[\sigma] \leq \frac{A[\sigma'] - A[\sigma]}{4}, \tag{60}$$

which is an expression of the quantum Bousso bound.

It is interesting to notice that the particular combinations in both sides of Eq. (60) are cutoff-independent, albeit the terms that comprise them are not.

Eq. (60) then tells us not only what the Bousso bound is, but also how the regularization of ΔS should be made. Instead of considering the entropy in the null surface connecting σ and σ', we are to consider the entropy in the half-Cauchy surfaces related to each surface and then compute their difference. This means that data far from the null hypersurface might also affect the entropy when the source is not clearly localized.

To recover the Bousso bound in its usual formulation, it is interesting for us to consider this setup with some amount of matter crossing the quantum lightsheet ("quantum" because $\Theta \leq 0$, instead of only $\theta \leq 0$). Consider then the schematics of Fig. 16a on the following page, where we start with a surface σ and end up at a surface σ' by following a quantum lightsheet. Since they are connected by a quantum lightsheet, we know $\Theta \leq 0$, and the QFC ensures $\Theta' \leq 0$. Hence, $A'_Q \leq A_Q$, which means

$$S'_{\text{out}} - S_{\text{out}} \leq \frac{A - A'}{4}. \tag{61}$$

Consider now what are S_{out} and S'_{out}. Some part of them should be comprised of the entropy S of the amount of matter crossing the quantum lightsheet. The rest, which lies in the half-Cauchy surfaces, we shall call S_{rest}. By looking at the diagram in Fig. 16a on the next page we find that

$$S'_{\text{out}} = S_{\text{rest}} + S, \tag{62}$$

$$S_{\text{out}} = S_{\text{rest}}. \tag{63}$$

Therefore, from Eq. (61), we find that

$$S \leq \frac{A - A'}{4}, \tag{64}$$

which is a statement of the Bousso bound.

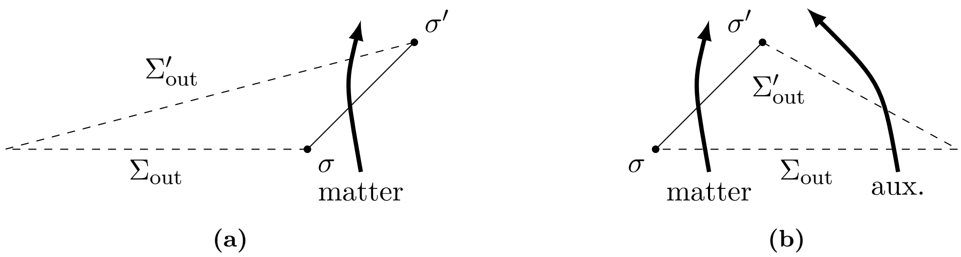


Figure 16: *Entropy flows through the setup considered in the QFC. In both cases, we start with a surface σ that evolves in time to a smaller surface σ' by following a quantum lightsheet. In Fig. 16a a single amount of matter crosses the quantum lightsheet connecting σ and σ' and crosses the half-Cauchy surface Σ'_{out} on its way out. In Fig. 16b the amount of matter crossing the quantum lightsheet is entangled to an auxiliary system such that the total entropy in Σ_{out} vanishes. However, only one of these entangled systems crosses the quantum lightsheet and only one crosses the half-Cauchy surface Σ'_{out}.*

Alternatively, we could consider the setup of Fig. 16b, in which there is an auxiliary system entangled to the matter system of interest so that the entropy at Σ_{out} vanishes. Once we consider the fact that both the matter system and the auxiliary system have the same entropy when they cross the quantum lightsheet or Σ'_{out} we conclude that, once again,

$$S \leq \frac{A - A'}{4}. \tag{65}$$

It is interesting to notice that $\theta \leq 0$ can typically enforced in GR by imposing the validity of the NEC. Can we do something similar in quantum settings where the NEC does not hold?

4 The Quantum Null Energy Condition

Let us now attempt to prove the QFC. In the process of doing so, we will find an interesting limit in which we recover an energy condition hereby called the quantum null energy condition. We begin by considering the off-diagonal QFC.

4.1 Off-Diagonal Quantum Focusing Conjecture

The QFC states that

$$\frac{\delta}{\delta V(y_2)} \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta A_Q}{\delta V(y_1)} \leq 0. \tag{66}$$

By "off-diagonal" we mean we will, for now, consider the case in which $y_1 \neq y_2$.

Notice that $A[V(y)]$ (the area of $V(y)$) is the integral of a local functional of $V(y)$. Therefore, we notice that, for $y_2 \neq y_1$,

$$\frac{\delta}{\delta V(y_2)} \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta A_Q}{\delta V(y_1)} = \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta}{\delta V(y_2)} \left(\frac{\delta A}{\delta V(y_1)} + 4 \frac{\delta S_{\text{out}}}{\delta V(y_1)} \right), \tag{67a}$$

$$= \frac{4}{\sqrt{g^V(y_1)}} \frac{\delta^2 S_{\text{out}}}{\delta V(y_2)\delta V(y_1)}. \tag{67b}$$

Therefore, in the off-diagonal case, the QFC becomes the statement that

$$\frac{\delta^2 S_{\text{out}}}{\delta V(y_2)\delta V(y_1)} \leq 0. \tag{68}$$

Hence, the off-diagonal QFC reduces to a statement of concavity of entropy, which is known to hold. Alternatively, Bousso, Fisher, Leichenauer, et al. (2016) show how the off-diagonal QFC reduces to the known statement that entropy satisfies strong subadditivity (Nielsen and Chuang 2011, Sec. 11.4).

4.2 Diagonal Quantum Focusing Conjecture

Consider now the case in which $y_1 = y_2$. Then let us first remark that

$$\Theta = \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta A_Q}{\delta V(y_1)}, \tag{69a}$$

$$= \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta A}{\delta V(y_1)} + \frac{4}{\sqrt{g^V(y_1)}} \frac{\delta S_{\text{out}}}{\delta V(y_1)}, \tag{69b}$$

$$= \frac{1}{\sqrt{g^V(y_1)}} \frac{\delta \sqrt{g^V(y_1)}}{\delta V(y_1)} + \frac{4}{\sqrt{g^V(y_1)}} \frac{\delta S_{\text{out}}}{\delta V(y_1)}, \tag{69c}$$

$$= \theta + \frac{4}{\sqrt{g^V(y_1)}} \frac{\delta S_{\text{out}}}{\delta V(y_1)}, \tag{69d}$$

where θ is the classical expansion. Notice that

$$\frac{\delta \sqrt{g^V(y_1)}}{\delta V(y_1)} = \frac{\delta A}{\delta V(y_1)} \tag{70}$$

because the only contribution to the derivative comes from the surface elements that actually depend on $V(y_1)$.

Using this expression, we find that

$$\frac{\delta \Theta}{\delta V(y_1)} = \frac{\delta \theta}{\delta V(y_1)} + \frac{\delta}{\delta V(y_1)} \left(\frac{4}{\sqrt{g^V(y_1)}} \frac{\delta S_{\text{out}}}{\delta V(y_1)} \right), \tag{71a}$$

$$= \frac{\delta \theta}{\delta V(y_1)} - \frac{4}{\sqrt{g^V(y_1)}\sqrt{g^V(y_1)}} \frac{\delta S_{\text{out}}}{\delta V(y_1)} \frac{\delta \sqrt{g^V(y_1)}}{\delta V(y_1)} + \frac{4}{\sqrt{g^V(y_1)}} \frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2}), \tag{71b}$$

$$= \frac{\delta \theta}{\delta V(y_1)} + \frac{4}{\sqrt{g^V(y_1)}} \left(\frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2} - \frac{\delta S_{\text{out}}}{\delta V(y_1)} \theta \right). \tag{71c}$$

Hence, the diagonal QFC becomes

$$\frac{\delta\Theta}{\delta V(y_1)} = \frac{\delta\theta}{\delta V(y_1)} - \frac{4}{\sqrt{g^V(y_1)}}\frac{\delta S_{\text{out}}}{\delta V(y_1)}\theta + \frac{4}{\sqrt{g^V(y_1)}}\frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2} \leq 0. \tag{72}$$

Using the Raychaudhuri equation we also find that

$$\Theta' = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{cd}k^c k^d - + \frac{4}{\sqrt{g^V(y_1)}}\left(\frac{4}{\sqrt{g^V(y_1)}}\frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2} - \frac{\delta S_{\text{out}}}{\delta V(y_1)}\theta\right), \tag{73}$$

where k^a is the null vector orthogonal to σ at y_1.

If we reinstate G and $\hbar$ and use the semiclassical Einstein equations, we get

$$\Theta' = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - 8\pi G \langle T_{cd} \rangle k^c k^d + \frac{4G\hbar}{\sqrt{g^V(y_1)}}\left(\frac{4}{\sqrt{g^V(y_1)}}\frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2} - \frac{\delta S_{\text{out}}}{\delta V(y_1)}\theta\right) \leq 0. \tag{74}$$

At any point in the manifold, we can build a congruence of null geodesics such that $\theta = 0$ and $\hat{\sigma}_{ab}\hat{\sigma}^{ab} = 0$. Hence, the diagonal QFC leads to a number of interesting results. In the $\hbar \to 0$ limit, we recover the NEC:

$$\langle T_{cd} \rangle k^c k^d \geq 0. \tag{75}$$

However, it is even more interesting to keep $\hbar$ finite. In this case, we find that

$$\langle T_{cd} \rangle k^c k^d \geq \frac{\hbar}{2\pi g^V(y_1)}\frac{\delta^2 S_{\text{out}}}{\delta V(y_1)^2}. \tag{76}$$

This is called the quantum null energy condition (QNEC), and is a natural generalization of the classical NEC. Notice that the dependency on G is canceled out, so that it turns out to be a statement about QFT itself, and in principle can be proved with much more effort within QFT only.

5 Further Reading

The main references we followed during this discussion were those by Bousso (1999a,b, 2002, 2018a,b,c, 2019). Nevertheless, it is useful recapitulating some of the historical references and further material.

The original papers that led to the discussion of black hole thermodynamics and the Bekenstein bound were due to Bardeen, Carter, and Hawking (1973), Bekenstein (1972, 1981), Hawking (1971), and Hawking (1975). Casini (2008) would later find a well-defined version of the Bekenstein bound.

Later, quantum versions of the Bousso bound would be proposed by Bousso, Casini, et al. (2014, 2015). The QFC was eventually introduced by Bousso, Fisher, Leichenauer, et al. (2016), and the first proof of the QNEC within QFT was given by Bousso, Fisher, Koeller, et al. (2016). Other, more general, proofs would later be given by Koeller and Leichenauer (2016) and Leichenauer, Levine, and Shahbazi-Moghaddam (2018).

Abbreviations

AdS anti-De Sitter

DEC dominant energy condition

EFEs Einstein field equations

FLRW Friedmann–Lemaître–Robertson–Walker

GR general relativity

GSL generalized second law

NEC null energy condition

QFC quantum focusing conjecture

QFT quantum field theory

QM quantum mechanics

QNEC quantum null energy condition

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A Notation and Conventions

We follow the notation and conventions used by Wald (1984), which corresponds to $+++$ in the Misner, Thorne, and Wheeler (1973) classification and employs abstract index notation.

While we often use units with $\hbar = c = G = k_B = 1$, we sometimes write these constants explicitly for clarity.