## Differential Geometry

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#### Abstract

These are some study notes I've been developing while studying elementary Differential Geometry. The text assumes familiarity with concepts on Topology, for this document, in particular, is a single chapter of a larger "book", aimed at the study of Hyperbolic Equations. You can find the "complete" work at http://fma.if.usp.br/~nickolas/ pdf/Hyperbolic_Equations.pdf.

I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments and etc through my e-mail (nickolas@fma.if.usp.br). If you wish to have a look at more works, please check my personal website http://fma. if.usp.br/~nickolas.

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## 1 Manifolds

This is a text on Mathematical Physics, and thus it could sound weird to talk about cartography in here. Nevertheless, I ask you to trust me that this discussion will lead us to some interesting concepts and consequences.

Our current goal is to make a good map of the Earth, which we shall consider as a perfect sphere*. This being a text on Mathematics, "good map" might sound somewhat vague, and thus we must be more specific.

Let us consider the unit 2-sphere given by

$$
\begin{equation*}
S^{2}\left\{x \in \mathbb{R}^{3} ;\|x\|=1\right\}, \tag{1.1}
\end{equation*}
$$

where the norm $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{3}$. Our goal is to find a homeomorphism $\varphi: S^{2} \rightarrow \mathbb{R}^{2}$. Such a function would allow us to map each point of the Earth's surface (here represented by $S^{2}$ ) to a single point in our map in a continuous way, so close points in the Earth are represented by close points in our map.

Unfortunately, this is an impossible task. $\mathrm{S}^{2}$ is the boundary of the set $\mathcal{B}_{1}(0)$ (the open ball centered at the origin with unitary radius), and we know that given a set $A$, it holds that $\partial \mathrm{A}=\overline{\partial \mathrm{A}}$, id est, the boundary of A is always closed. Furthermore, $\mathrm{S}^{2} \subseteq \mathcal{B}_{2}(0)$, meaning it is a bounded set. As a closed and bounded set, the Heine-Borel Theorem ensures it is a compact set.

On the other side, $\mathbb{R}^{2}$ is open, but it is not bounded. Hence, the Heine-Borel Theorem guarantees $\mathbb{R}^{2}$ is not a compact set.

Since compactness is preserved by homeomorphisms, it is impossible to obtain a homeomorphism between a compact set such as $S^{2}$ and a non-compact set such as $\mathbb{R}^{2}$. Thus, there is no such thing as a perfect map.

In the absence of a perfect map, we must content ourselves with simpler versions. What is our possible option?

Cartographers might not be able to produce a single map of the entire Earth at once which happens to be a homeomorphism, but it is still possible to map the Earth. We have two options: either we abandon our requirement of continuity - and then we may get something as the Mercator projection, which maps the entire Earth but is not continuous everywhere - or we may abandon the idea of mapping the entire Earth at once and stick

[^0]to continuity. This second option would still allow us to make an atlas if we are careful enough. A single chart can't describe the whole Earth at once, but maybe 12 charts can.

In this text, we are interested in studying the case in which we stick to continuity and abandon the desire of mapping the whole Earth at once. The other possibility we leave for the cartographers to explore.

Our goal now is to obtain a way of finding a certain amount of "charts" $(\mathrm{U}, \varphi)$, where U is an open set in $S^{2}$ and $\varphi: \mathrm{U} \rightarrow \operatorname{Ran} \varphi \subseteq \mathbb{R}^{2}$ is a homeomorphism (notice this implies $\operatorname{Ran} \varphi$ is an open set in $\mathbb{R}^{2}$, since $\varphi$ is an open map), such that the collection of all such charts covers $\mathrm{S}^{2}$ as a whole.

Example [Charting S ${ }^{2}$ ]:
This requirement is easy to be fulfilled Let us split $S^{2}$ in six parts given by

$$
\begin{equation*}
\mathrm{u}_{\mathrm{i}}^{ \pm} \equiv\left\{\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right) \in \mathrm{S}^{2} ; \operatorname{sign}\left(\mathrm{x}_{\mathrm{i}}\right)= \pm 1\right\} . \tag{1.2}
\end{equation*}
$$

These sets are open, for they are of the form

$$
\begin{equation*}
\mathrm{U}_{1}^{+}=\mathrm{S}^{2} \cap\left\{\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right) \in \mathbb{R} ; \mathrm{x}_{1}>0\right\}, \tag{1.3}
\end{equation*}
$$

id est, they are the intersection of the subspace we are considering $S^{2}$ with an open set in the ambient space $\mathbb{R}^{3}$.

These sets do cover $S^{2}$. Let $\left(x_{1}, x_{2}, x_{3}\right) \in S^{2}$. If $\left(x_{1}, x_{2}, x_{3}\right) \notin U_{1}^{+} \cup U_{1}^{-}$, then it must hold that $x_{1}=0$. If $\left(x_{1}, x_{2}, x_{3}\right) \notin \mathrm{U}_{2}^{+} \cup \mathrm{U}_{2}^{-}$, then it must hold that $x_{2}=0$. Since $x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1$, this implies $x_{3}= \pm 1$, and hence $\left(x_{1}, x_{2}, x_{3}\right) \in U_{3}^{+} \cup U_{3}^{-}$.

We may now consider the maps $\varphi_{1}^{ \pm}: \mathrm{U}_{\mathrm{i}}^{ \pm} \rightarrow \mathcal{B}_{1}(0)$ given by $\varphi_{1}^{ \pm}\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{2}, x_{3}\right)$ with similar definitions for $\varphi_{i}^{ \pm}$. We shall prove $\varphi_{1}^{ \pm}$is a homeomorphism. The remaining cases are similar.

Let us begin by proving $\varphi_{1}^{+}$is invertible. $\varphi_{1}^{+}$is onto. Indeed, given $\left(x_{2}, x_{3}\right) \in \mathcal{B}_{1}(0)$, it holds that

$$
\begin{equation*}
\left(\sqrt{1-x_{2}^{2}-x_{3}^{2}}, x_{2}, x_{3}\right) \in \mathrm{U}_{1}^{+}, \tag{1.4}
\end{equation*}
$$

since

$$
\begin{align*}
\sqrt{1-x_{2}^{2}-x_{3}^{2}}
\end{align*}
$$

and $\sqrt{1-x_{2}^{2}-x_{3}^{2}}>0$. Notice that

$$
\begin{equation*}
\varphi_{1}^{+}\left(\left(\sqrt{1-x_{2}^{2}-x_{3}^{2}}, x_{2}, x_{3}\right)\right)=\left(x_{2}, x_{3}\right) . \tag{1.6}
\end{equation*}
$$

Furthermore, $\varphi_{1}^{+}$is one-to-one. Suppose $\left(y, x_{2}, x_{3}\right),\left(z, x_{2}, x_{3}\right) \in U_{1}^{+}$are such that

$$
\begin{equation*}
\varphi_{1}^{+}\left(\left(y, x_{2}, x_{3}\right)\right)=\varphi_{1}^{+}\left(\left(z, x_{2}, x_{3}\right)\right)=\left(x_{2}, x_{3}\right) . \tag{1.7}
\end{equation*}
$$

Then notice that

$$
\begin{gather*}
z^{2}+x_{2}^{2}+x_{3}^{2}=1 \\
z^{2}=1-x_{2}^{2}-x_{3}^{2} \\
z=+\sqrt{1-x_{2}^{2}-x_{3}^{2}} \tag{1.8}
\end{gather*}
$$

where the last step used the fact that $z>0$. The same argument applies to $y$, and hence $y=z$ and we conclude $\left(y, x_{2}, x_{3}\right)=\left(z, x_{2}, x_{3}\right)$. Hence, $\varphi_{1}^{+}$is bijective.

We now must prove $\varphi_{1}^{+}$and its inverse are continuous. Notice that the components of $\varphi_{1}^{+}$are simply the projections $\pi_{i}\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=x_{i}$ (which are continuous when $\mathbb{R}^{3}$ is equipped with the product topology, and it is) restricted to $\mathrm{U}_{1}^{+}$, and hence they are continuous in the relative topology.

The inverse is also continuous and one can prove it by employing the fact that a function $f: Y \rightarrow X_{\lambda \in \Lambda} X_{\lambda}$ is continuous if, and only if, all the coordinate functions ( $\pi_{\lambda} \circ f$ ): $Y \rightarrow X_{\lambda}$ are continuous.

Notice that $\left(\pi_{2} \circ \varphi_{1}^{+-1}\right)\left(x_{2}, x_{3}\right)=x_{2}$ with a similar expression for $\pi_{3}$ and $\left(\pi_{2} \circ\right.$ $\left.\varphi_{1}^{+-1}\right)\left(x_{2}, x_{3}\right)=\sqrt{1-x_{2}^{2}-x_{3}^{2}}$. Thus, $\left(\pi_{i} \circ \varphi_{1}^{+-1}\right)$ are simply projections restricted to a certain domain for $\mathfrak{i}=1,2 .\left(\pi_{1} \circ \varphi_{1}^{+-1}\right)$ is a composition of continuous functions, and I leave for you the task of proving it. Therefore, we see that $\varphi_{1}^{+}$is indeed a homeomorphism.

This construction allowed us to chart $S^{2}$ in parts. By diving the Earth in six pieces, we can chart each piece continuously. Notice that the charts we chose superpose: for example, $(1,1,1) \in U_{i}^{+}$for $i=1,2,3$. This means we should expect some agreement between the different charts: I must be able to change from a chart to another one continuously. In cartographical terms, suppose you are following the trajectory of a ship in one of the charts of an atlas. If the trajectory reaches the edge of the page, you must be able to keep following it in another page without any contradictions. If the trajectory was depicted in a certain way in a page, it must be descripted in an analogue way in another page.

Let us put it in mathematical terms: if you have two charts $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ such that $\mathrm{U} \cap \mathrm{V} \neq \varnothing$, you would like to be able to continuously transition from $\varphi(\mathrm{U} \cap \mathrm{V})$ to $\psi(\mathrm{U} \cap \mathrm{V})$. Is this possible?

Indeed it is. Let us write $\varphi(\mathrm{U} \cap \mathrm{V})=\mathrm{A}$ and $\psi(\mathrm{U} \cap \mathrm{V})=\mathrm{B}$. Notice that we have a function $\left(\varphi \circ \psi^{-1}\right): B \rightarrow A$ which is a homeomorphism, since it is the composition of two homeomorphisms. Thus. we can transit continuously between the pages of our atlas. We have completely described the Earth in the pages of a book in a continuous manner, despite being unable to produce a unique map describing the entire planet in a continuous way.

This might solve the problem for the cartographers, but this is a text in Mathematical Physics. The next question a mathematician would ask might be something similar to "Ok, but what if the Earth was a torus?". Can we make this more general? If so, how general?

In order to address these questions, we should step back a bit and provide precise definitions for the concepts we've developed so far.

Definition 1 [Locally Euclidean Space]:
Let $(M, \tau)$ be a topological space. We say it is a locally Euclidean space of dimension $n$ if, and only if, every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open subset of $\mathbb{R}^{n}$. The pair $(\mathrm{U}, \varphi)$ is said to be a chart, U is said to be a coordinate neighborhood and $\varphi$ is said to be a coordinate system on U . If $\varphi(\mathrm{p})=0$, the chart $(\mathrm{U}, \varphi)$ is said to be centered at $p$.

## Theorem 2:

Let $(M, \tau)$ be a topological space. $(M, \tau)$ is a locally Euclidean space if, and only if, every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open ball of $\mathbb{R}^{n}$.

Proof:
Suppose every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open ball of $\mathbb{R}^{n}$. Since every open ball is an open set, it follows immediately that $(M, \tau)$ is locally Euclidean.

Suppose $(M, \tau)$ is locally Euclidean. Let $p \in M$. We know there is a pair $(U, \varphi)$ such that U is an open neighborhood of p and $\varphi: \mathrm{U} \rightarrow \varphi(\mathrm{U})$ is a homeomorphism. Since $\varphi$ is a homeomorphism, $\varphi(\mathrm{U})$ is an open set. In particular, it is an open neighborhood of $\varphi(\mathfrak{p})$. Therefore, there is some $\epsilon>0$ such that $\varphi(\mathfrak{p}) \subseteq \mathcal{B}_{\epsilon}(\varphi(\mathfrak{p})) \subseteq \varphi(\mathrm{U})$. Furthermore, since $\varphi$ is a homeomorphism, $\varphi^{-1}\left(\mathcal{B}_{\epsilon}(\varphi(p))\right)$ is an open neighborhood of $p$. Also, the restriction of $\varphi$ to $\varphi^{-1}\left(\mathcal{B}_{\epsilon}(\varphi(p))\right)$ is a homeomorphism, proving that $p \in M$ has an open neighborhood which has an homeomorphism onto an open ball of $\mathbb{R}^{n}$.

## Definition 3 [Atlas]:

Let $(M, \tau)$ be a locally Euclidean space of dimension $n$. An atlas on $(M, \tau)$ is a collection $\mathcal{A}=\left\{\left(\mathrm{U}_{\lambda}, \varphi_{\lambda}\right)\right\}_{\lambda \in \Lambda}$ of charts on $(M, \tau)$ such that $M=\bigcup_{\lambda \in \Lambda} U_{\lambda}$.

Notice that, by definition, every locally Euclidean spaces admits at least one atlas.

## Definition 4 [Topological Manifold]:

A topological manifold of dimension $\mathfrak{n}$ is a Hausdorff, second-countable, locally Euclidean space of dimension $n$.

In order to prove the well-definition of the dimension of a topological manifold, we shall employ (without proof) the Theorem of Topological Invariance of Dimension, which is Corollary 1.6.3 of [12].

Theorem 5 [Topological Invariance of Dimension]:
Let $\mathrm{n}, \mathrm{m} \in \mathbb{N}, \mathrm{n}>\mathrm{m}$. Let $\varnothing \neq \mathrm{U} \subseteq \mathbb{R}^{n}$. There is no continuous injective mapping from U to $\mathbb{R}^{m}$. In particular, $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ are not homeomorphic.

The following result will also be useful:

## Lemma 6:

Let $\mathrm{n} \in \mathbb{N}, \mathrm{p} \in \mathbb{R}^{n}, \epsilon>0 . \mathcal{B}_{\epsilon}(\mathfrak{p})$ is homeomorphic to $\mathbb{R}^{n}$.
Proof:
Firstly we notice that $\mathcal{B}_{\epsilon}(\mathfrak{p})$ is homeomorphic to $\mathcal{B}_{1}(0)$. This can be proven in a simple
way by employing the fact that $\mathbb{R}^{n}$ is a locally convex space. More details can be found at [1].

We now must simply prove that $\mathcal{B}_{1}(0)$ and $\mathbb{R}^{n}$ are homeomorphic. Consider the function $\varphi: \mathcal{B}_{1}(0) \rightarrow \mathbb{R}^{n}$

$$
\begin{equation*}
\varphi(x)=\tan \left(\frac{\pi\|x\|}{2}\right) x \tag{1.9}
\end{equation*}
$$

It is a composition of continuous functions, since tan is continuous on the interval $[0,1)$, the norm is continuous on the topology induced by itself and product by a scalar is continuous on $\mathbb{R}^{n}[1]$. It is also a inversible function with continuous inverse. Hence, it is a homeomorphism.

## Theorem 7:

The dimension of a topological manifold is well-defined.
Proof:
Let $(M, \tau)$ be a topological manifold of dimension $n$ and assume, for the sake of contradiction, that it is also a topological manifold of dimension $m \neq n$. We assume without any loss of generality that $n>m$.

Let $p \in M$. Due to Theorem 2, we know there is an open set $U$ with $p \in U$ and a homeomorphism $\varphi: U \rightarrow \mathcal{B}_{\epsilon}(x) \subseteq \mathbb{R}^{n}$, for some $\epsilon>0$ and some $x \in \mathbb{R}^{n}$. Similarly, there is an open set $V$ with $p \in V$ and a homeomorphism $\psi: V \rightarrow \psi(V) \subseteq \mathcal{B}_{\delta}(y)$, for some $\delta>0$ and some $y \in \mathbb{R}^{m}$. Due to Lemma 6 , we know that $V$ is homeomorphic to $\mathbb{R}^{m}$ (let's call this homeomorphism $g$ ) and $U$ is homeomorphic to $\mathbb{R}^{n}$ (let's call this homeomorphism $f$ ).

We may consider the open set $U \cap V$. We know that $f: U \cap V \rightarrow f(U \cap V) \subseteq \mathbb{R}^{n}$ is a homeomorphism and so is $g: U \cap V \rightarrow g(U \cap V) \subseteq \mathbb{R}^{m}$. Hence, $\left(g \circ f^{-1}\right): f(U \cap V) \rightarrow \mathbb{R}^{m}$ is a continuous injective map. Theorem 5 tells us this is a contradiction. Hence, $(M, \tau)$ can't have two different dimensions, proving the dimension of $M$ is well-defined.

## Notation:

We denote the dimension of a topological manifold $(M, \tau)$ by $\operatorname{dim} M$.
The Hausdorff and second-countability properties are demanded in order to add some structure to the manifold. It is always interesting to have uniqueness of limits (which is provided by the Hausdorff condition), especially considering we will eventually develop a generalization of Calculus on structures similar to (but more complicated than) these. Second-countability also will allow us to obtain more results (at the cost of generality), but in particular it will prove its importance later on, when we start dealing with partitions of unity.

The last requirement could be considered the soul of our interest: develop a theory of structures which locally resemble $\mathbb{R}^{n}$, which we are already familiar with. The notions we shall develop are closely tied to the ideas of using charts and atlases to map the Earth: you don't need to know the actual Earth if your atlas is good enough. Being able to read the information in the charts will be enough to obtain information about the real world. We may illustrate this in the following proposition.

## Definition 8 [Continuous Curve]:

Let $(M, \tau)$ be a topological manifold. A curve on $M$ is a function $\gamma: I \rightarrow M$, where $\mathrm{I} \subseteq \mathbb{R}$. A curve is said to be continuous if it is continuous as a function between topological spaces, where $\mathbb{R}$ is considered to be equipped with the standard topology. Continuity of $\gamma$ at a point $\lambda \in \mathbb{R}$ is defined in a similar manner.

## Proposition 9:

Let $(M, \tau)$ be a topological manifold of dimension $n$. Consider a curve $\gamma: \mathbb{R} \rightarrow M$. Let $\lambda \in \mathbb{R}$ and consider a chart $(\mathrm{U}, \varphi)$ of M such that $\gamma(\lambda) \in \mathrm{U} . \gamma$ is continuous at $\lambda$ if, and only if, $\varphi \circ \gamma$ is continuous at $\lambda$.

Proof:
Suppose $\gamma$ is continuous at $\lambda$. Since $\varphi$ is a homeomorphism, $\varphi \circ \gamma$ is a composition of continuous functions at $x$ and hence it is continuous. On the other hand, if $\varphi \circ \gamma$ is continuous at $\chi$, notice that $\gamma=\varphi^{-1} \circ(\varphi \circ \gamma)$, and hence $\gamma$ is the composition of continuous functions.

We may illustrate this notion in the following commutative diagram*.


One should notice the fact that even though we may find whether $\gamma$ is continuous by looking at $\varphi \circ \gamma$, the fact that $\gamma$ is continuous at a point $\lambda$ does not depend on the chart we choose. Suppose, for example, that $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ are charts with $\gamma(\lambda) \in \mathrm{U} \cap \mathrm{V}$. Then $\varphi \circ \gamma$ is continuous at $\lambda$ if, and only if, $\psi \circ \gamma$ is continuous at $\lambda$. A way of noticing it is by looking at the map $\psi \circ \varphi^{-1}$ (which is a composition of continuous maps, and therefore is continuous as well):

$$
\begin{align*}
\psi \circ \gamma & =\psi \circ\left(\varphi^{-1} \circ \varphi\right) \circ \gamma, \\
& =\left(\psi \circ \varphi^{-1}\right) \circ(\varphi \circ \gamma) . \tag{1.10}
\end{align*}
$$

Since $\psi \circ \varphi^{-1}$ is continuous, continuity of $\varphi \circ \gamma$ implies continuity of $\psi \circ \gamma$.
This remark might seem pointless, since we have already proven that continuity of $\varphi \circ \gamma$ at $x$ for any chart $(\mathrm{U}, \varphi)$ containing $\gamma(x)$ is equivalent to continuity of $\gamma$ itself at $x$, but this notion will be useful when we try to generalize these concepts, which is the reason we shall put these remarks in a more formal setting.

## Definition 10 [Chart Transition Maps]:

Let $(M, \tau)$ be a topological manifold of dimension $n$ and let $(U, \varphi)$ and $(V, \psi)$ be charts on $M$ such that $U \cap V \neq \varnothing$. The chart transition maps, or simply transition maps or transition

[^1]functions, between $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ are the maps
\[

$$
\begin{equation*}
\varphi \circ \psi^{-1}: \psi(\mathrm{U} \cap \mathrm{~V}) \rightarrow \varphi(\mathrm{U} \cap \mathrm{~V}), \quad \psi \circ \varphi^{-1}: \varphi(\mathrm{U} \cap \mathrm{~V}) \rightarrow \psi(\mathrm{U} \cap \mathrm{~V}) . \tag{1.11}
\end{equation*}
$$

\]

## Lemma 11:

Let $(\mathrm{M}, \tau)$ be a topological manifold of dimension n and let $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ be charts on M such that $\mathrm{U} \cap \mathrm{V} \neq \varnothing$. The chart transition maps between $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ are continuous.
Proof:
Both $\varphi$ and $\psi$ are homeomorphisms, and hence both them and their inverses are continuous. Thus, the composition of any combination of $\psi, \psi^{-1}, \varphi$ and $\varphi^{-1}$ is a continuous function.

This provides us with a theory of spaces which resemble $\mathbb{R}^{n}$ in terms of continuity of functions. If we want to check what was the real trajectory of a ship on the topological manifold, we can simply see the trajectory across the charts (and use the continuous chart transition maps to "flip pages" on the atlas) and evaluate continuity by looking at this projection.

However, what if we not only wanted to know the ship's trajectory, but also its velocity? As we know from elementary Physics, this requires a theory of differentiation, which we do not possess for such general spaces, since Topology can only deal with continuity.

Nevertheless, we have already seen a possible way of defining differentiability of curves in this context. Proposition 9 states that we can speak of continuity without needing to pay attention to the manifold's topology, so imposing a similar result could provide a satisfactory notion of differentiability.

However, a problem arises. We must guarantee that our definition is chart-independent, since it should reflect a property of the curve itself, and not a property of the curve's projection through a specific chart. This can be visualized in the following diagram:


If we want to give a proper definition of differentiability at $p \in U \cap V$ to $\gamma$ by analysing, say, whether $\psi \circ \gamma$ is differentiable, then the same result should be obtained by analysing $\varphi \circ \gamma$.

Therefore, we must have that, given charts $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ with $p \in \mathrm{U} \cap \mathrm{V}$, then $\psi \circ \gamma$ is differentiable at $p$ if, and only if, $\varphi \circ \gamma$ is differentiable at $p$.

The diagram tells us how to achieve this: $\psi \circ \gamma=\left(\psi \circ \varphi^{-1}\right) \circ(\varphi \circ \gamma)$. If $\varphi \circ \gamma$ is differentiable and $\psi \circ \varphi^{-1}$ is differentiable, then $\psi \circ \gamma$ will also be differentiable. This motivates the definition of compatible charts.

Definition 12 [ ${ }^{\mathrm{k}}$-compatible Charts]:
Let $(M, \tau)$ be a locally Euclidean space. Let $(U, \varphi)$ and $(V, \psi)$ be charts on $(M, \tau)$. The charts are said to be $\mathcal{C}^{\mathrm{k}}$-compatible if, and only if, either of the following requirements hold:
i. $\mathrm{U} \cap \mathrm{V}=\varnothing$;
ii. $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are of class $\mathcal{C}^{k}$.

Definition 13 [ ${ }^{\mathrm{k}}$-atlas]:
Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be an atlas on $(M, \tau) . \mathcal{A}$ is said to be a $\mathcal{C}^{\mathrm{k}}$-atlas if, and only if, the charts on $\mathcal{A}$ are pairwise $\mathcal{C}^{\mathrm{k}}$-compatible.

In particular, $C^{\infty}$-atlases are commonly referred to as smooth atlases.
Curiously, in order to define differentiability, we are not asking for more structure. We are asking for less.

The problem we had with our original topological manifold was that some transition maps were not $\mathcal{C}^{1}$-compatible, and thus it was not possible to define differentiability (which we shall do soon) in a chart independent manner. We simply had too many pages on our atlas and some of them were kind of weird when we speak about differentiability. Our solution was to tear off these bad pages and keep a smaller, but more powerfull, atlas.

On the other hand, after we tear off the pages of our atlas, we'll be careful to keep all the useful pages. In other words, we will ask for our manifolds to be equipped with a maximal atlas.

Definition 14 [Maximal $\mathcal{C}^{\mathrm{k}}$-atlas]:
Let $(M, \tau)$ be a topological manifold and let $\mathcal{A}$ be a $\mathcal{C}^{k}$-atlas on $(M, \tau)$. $\mathcal{A}$ is said to be maximal if, and only if, for every $\mathcal{C}^{\mathrm{k}}$-atlas $\mathcal{A}^{\prime}$ with $\mathcal{A} \subseteq \mathcal{A}$ it holds that $\mathcal{A}=\mathcal{A}^{\prime}$.

A maximal $\mathcal{C}^{\mathrm{k}}$-atlas on a topological manifold $(M, \tau)$ is also referred to as a $\mathcal{C}^{\mathrm{k}}$-structure on $(M, \tau)$. Once again, the $k=\infty$ case is referred commonly as "smooth" instead of $e^{\infty}$.

Definition 15 [ $\mathrm{C}^{\mathrm{k}}$-manifold]:
Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^{\mathrm{k}}$-atlas on $(M, \tau)$. The triple $(M, \tau, \mathcal{A})$ is said to be a $\left.\mathcal{C}^{[ } k\right]$-manifold.

In particular, $\mathfrak{C}^{\infty}$-manifolds are commonly referred to as smooth manifolds or differentiable manifolds.

Remark:
Notice that a $\mathcal{C}^{k}$-manifold is a $\mathcal{C}^{l}$-manifold for every $l \leqslant k$.
Requiring a maximal atlas might seem silly, but it is well justified: it comes for free if you already have any other atlas.

## Lemma 16:

Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathrm{C}^{\mathrm{k}}$-atlas on $(\mathrm{M}, \tau)$. Let $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ be charts on $(\mathrm{M}, \tau)$. If both $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ are compatible with the atlas $\mathcal{A}$, then they are compatible with each other.

Proof:
If $\mathrm{U} \cap \mathrm{V}=\varnothing$, the proof is complete. Let us then assume $\mathrm{U} \cap \mathrm{V} \neq \varnothing$.
$\mathcal{A}$ covers $M$, and therefore, given $p \in U \cap V$, there is some chart $(W, \chi)$ with $p \in W$. By hypothesis, $(W, \chi)$ is compatible with both $(U, \varphi)$ and $(V, \psi)$. We may represent this in the diagram


Since $(W, \chi)$ is compatible with both $(U, \varphi)$ and $(V, \psi)$, we know that $\chi \circ \psi^{-1}$ is $\mathcal{C}^{k}$ at $\psi(\mathrm{U} \cap \mathrm{V} \cap \mathrm{W})$ and $\varphi \circ \chi^{-1}$ is $\mathcal{C}^{\mathrm{k}}$ at $\chi(\mathrm{U} \cap \mathrm{V} \cap \mathrm{W})$. Hence, $\varphi \circ \psi^{-1}$ is $\mathrm{C}^{\mathrm{k}}$ at $\psi(\mathrm{U} \cap \mathrm{V} \cap \mathrm{W})$ and, in particular, at $\psi(p)$. Since $p \in U \cap V$ was arbitrary, we see that $\varphi \circ \psi^{-1}$ is $\mathcal{C}^{k}$ at $\psi(\mathrm{U} \cap \mathrm{V})$. A similar argument proves that $\psi \circ \varphi^{-1}$ is $\mathcal{C}^{k}$ at $\varphi(\mathrm{U} \cap \mathrm{V})$. Therefore, $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ are $\mathcal{C}^{\mathrm{k}}$-compatible.

## Proposition 17:

Let $(\mathrm{M}, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathrm{C}^{\mathrm{k}}$-atlas on $(\mathrm{M}, \tau) . \mathcal{A}$ is contained on a unique maximal $\mathrm{C}^{\mathrm{k}}$-atlas.
Proof:
Consider the set $\overline{\mathcal{A}}$ of all charts $\mathcal{C}^{\mathrm{k}}$-compatible with $\mathcal{A}$. Notice that $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and, as a consequence, $\overline{\mathcal{A}}$ is an atlas, for it is a collection of charts that covers $M$. We must now prove that it is a $\mathrm{C}^{\mathrm{k}}$-atlas and that it is maximal.

Let $(\mathrm{U}, \varphi),(\mathrm{V}, \psi) \in \overline{\mathcal{A}}$. By hypothesis, both of them are $\mathcal{C}^{\mathrm{k}}$-compatible with $\mathcal{A}$ and, due to Lemma 16, are compatible with each other. Therefore, $\overline{\mathcal{A}}$ is a $\mathcal{C}^{\mathrm{k}}$-atlas.

Suppose now $\mathcal{A}^{\prime}$ is a $\mathcal{C}^{\mathrm{k}}$-atlas containing $\overline{\mathcal{A}}$. Notice $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \mathcal{A}^{\prime}$. Thus, every chart $(\mathrm{U}, \varphi)$ in $\mathcal{A}^{\prime}$ is $\mathcal{C}^{\mathrm{k}}$-compatible with $\mathcal{A}$. Thus, by definition of $\overline{\mathcal{A}}$, every chart $(\mathrm{U}, \varphi)$ of $\mathcal{A}^{\prime}$ is in $\overline{\mathcal{A}}$, id est, $\mathcal{A}^{\prime} \subseteq \overline{\mathcal{A}}$. Therefore, $\overline{\mathcal{A}}=\mathcal{A}^{\prime}$, proving $\overline{\mathcal{A}}$ is maximal.

Finally, we must prove $\overline{\mathcal{A}}$ is unique. Suppose $\mathcal{A}^{\prime}$ is some $\mathcal{C}^{\mathrm{k}}$-atlas with $\mathcal{A} \subseteq \mathcal{A}^{\prime}$. Then every chart in $\mathcal{A}^{\prime}$ is compatible with $\mathcal{A}$ and hence $\mathcal{A}^{\prime} \subseteq \overline{\mathcal{A}}$, so either $\mathcal{A}^{\prime}=\overline{\mathcal{A}}$ or $\mathcal{A}^{\prime}$ is not maximal. One way or the other, the proof is complete.

Proposition 17 guarantees that, when proving some topological space is a $\mathcal{C}^{k}$-manifold, we do not need to bother with describing the whole maximal atlas. Instead, it suffices to find some atlas and the existence of a maximal atlas is guaranteed.

A result due to Hassler Whitney states that, for every $k>0$, a maximal $\mathcal{C}^{k}$-atlas contains a smooth atlas[6]. As a consequence, we will be mostly interested on the theory of smooth manifolds.

The restriction $k \neq 0$ is important: there are examples of topological manifolds that do not admit a smooth structure. The first example[7] of such a manifold is a 10 -dimensional manifold constructed by Michel Kervaire in 1960[5].

Let us check a few examples of manifolds.

## Example [Euclidean Space]:

The first example of smooth manifold one might consider is $\mathbb{R}^{n}$ itself, which is a Hausdorff, second-countable space. An atlas is given by $\left\{\left(\mathbb{R}^{n}, \mathrm{id}\right)\right\}$, where id: $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is the function that maps $\chi \mapsto x$.

Example [Locally Euclidean space which is not Hausdorff]:
A simple example of a locally euclidean space which is not Hausdorff is the line with two origins: the real line with an extra point.

We begin by picking some set which we already know to exist. As any set will do, let $\sim$ denote a leaf. We write $X=\mathbb{R} \cup\{\sim\}$.

We now proceed to define a topology in $X$. Let $\mathfrak{B}_{\mathbb{R}}$ be the basis of open intervals for the standard topology in $\mathbb{R}$. Let $\mathfrak{B}_{\infty} \equiv\left\{\{\boldsymbol{\omega}\} \cup \mathrm{B} \backslash\{0\} ; \mathrm{B} \in \mathfrak{B}_{\mathbb{R}}\right\}$. We define $\mathfrak{B} \equiv \mathfrak{B}_{\mathbb{R}} \cup \mathfrak{B}_{\infty}$. $\mathfrak{B}$ is a basis for a non-Hausdorff topology in $X$. On the other hand, every point $p$ has an open neighbourhood which can be mapped with the identity (or with a quasi-identity $x \mapsto x$ for $x \neq \infty$ and $\omega \mapsto 0$ ) to $\mathbb{R}$. Thus, it is locally Euclidean.

Example [2-sphere]:
The construction made on the beginning of this chapter can be used to prove that $S^{2}$ is a smooth manifold.

Definition 18 [Surface in $\mathbb{R}^{3}$ ]:
Let $S \subseteq \mathbb{R}^{3}$. S is said to be a surface if, and only if, for every $p \in S$ there is an open neighborhood $V_{p}$ of $p$ in $\mathbb{R}^{3}$, an open subset $U_{p}$ of $\mathbb{R}^{2}$, and a smooth map $f_{p}: U_{p} \rightarrow \mathbb{R}^{3}$ such that $S \cap V_{p}$ is the graph of $z=f_{p}(x, y), x=f_{p}(y, z)$ or $y=f_{p}(z, x)$.

## Proposition 19:

Let $\mathrm{S} \subseteq \mathbb{R}^{3}$ be a surface. For every $\mathrm{p} \in \mathrm{S}$, let $\mathrm{V}_{\mathrm{p}}$ be an open neighborhood of p in $\mathbb{R}^{3}, \mathrm{u}_{\mathrm{p}}$ be an open subset of $\mathbb{R}^{2}$, and $\mathrm{f}_{\mathrm{p}}: \mathrm{U}_{\mathrm{p}} \rightarrow \mathbb{R}^{3}$ be a smooth map such that $\mathrm{S} \cap \mathrm{V}_{\mathrm{p}}$ is the graph of $z=f_{p}(x, y), x=f_{p}(y, z)$ or $y=f_{p}(z, x) .(S, \tau, \mathcal{A})$ is a smooth manifold, where $\tau$ is the relative topology of $S$ with respect to $\mathbb{R}^{3}$ and $\mathcal{A}$ is the maximal smooth atlas associated to the smooth atlas

$$
\begin{equation*}
\mathcal{A}=\left\{\left(S \cap V_{p}, \varphi_{p}\right) ; p \in M\right\} \tag{1.12}
\end{equation*}
$$

where the maps $\varphi_{p}$ are defined such that $(x, y, z) \mapsto(x, y)$ when $S \cap V_{p}$ is the graph of $z=f_{p}(x, y)$, with similar definitions for the other cases.
Proof:
Since $\mathbb{R}^{3}$ is Hausdorff and second-countable, so is $(S, \tau)$.
Pick $p \in S$. We know there is an open neighborhood $V_{p}$ of $p$ in $\mathbb{R}^{3}$, an open subset $U_{p}$ of $\mathbb{R}^{2}$, and a smooth map $f_{p}: U_{p} \rightarrow \mathbb{R}^{3}$ such that $S \cap V_{p}$ is the graph of $z=f_{p}(x, y)$, $x=f_{p}(y, z)$ or $y=f_{p}(z, x)$. Notice that $S \cap V_{p}$ is an open neighborhood of $p$ in $(S, \tau)$.

Let us assume, without any loss of generality, that $f_{p}: U_{p} \rightarrow \mathbb{R}^{3}$ is such that $S \cap V_{p}$ is the graph of $z=f_{p}(x, y)$. Then $S \cap V_{p}=\left\{\left(x, y, f_{p}(x, y)\right) \in \mathbb{R}^{3} ;(x, y) \in U_{p}\right\}$.

Let $\varphi_{p}: S \cap V_{p} \rightarrow \varphi_{p}\left(S \cap V_{p}\right)$ be given by $\varphi_{p}((x, y, z))=(x, y)$. One may show this function is bijective and continuous (it is a projection). The inverse is $\varphi_{p}^{-1}((x, y))=$ $\left(x, y, f_{p}(x, y)\right)$. Since $f_{p}$ is a smooth map, this is $j u s t ~ a ~ c o m p o s i t i o n ~ o f ~ c o n t i n u o u s ~ f u n c t i o n s . ~$ Hence, $\varphi_{p}$ is a homeomorphism and we have proven $(S, \tau)$ is a topological manifold. The homeomorphisms are onto open sets of $\mathbb{R}^{2}$, and thus $\operatorname{dim} S=2$.

We must now prove $\mathcal{A}$ is a smooth atlas. It surely is an atlas, for $\forall p \in M, \exists\left(S \cap V_{p}, \varphi_{p}\right) \in$ $\mathcal{A}$ with $p \in S \cap V_{p}$.

Let $\mathrm{p}, \mathrm{q} \in \mathrm{M}$ such that $\mathrm{S} \cap \mathrm{V}_{\mathrm{p}} \cap \mathrm{V}_{\mathrm{q}} \neq \varnothing$ (if the intersection vanishes, the result is trivial). We assume, without any loss of generality, that $S \cap V_{p}=\left\{\left(x, y, f_{p}(x, y)\right) \in \mathbb{R}^{3} ;(x, y) \in U_{p}\right\}$ and $S \cap V_{q}=\left\{\left(x, f_{q}(z, x), z\right) \in \mathbb{R}^{3} ;(z, x) \in U_{q}\right\}$. We want to prove that $\varphi_{p} \circ \varphi_{q}^{-1}$ is smooth (the proof already applies to the other transition function by simply exchanging $p$ and $q$ ).

Notice that $\varphi_{q}^{-1}((z, x))=\left(x, f_{q}(z, x), z\right)$ and $\varphi_{p}((x, y, z))=(x, y)$. Hence,

$$
\begin{align*}
\left(\varphi_{p} \circ \varphi_{q}^{-1}\right)((z, x)) & =\varphi_{p}\left(\left(x, f_{q}(z, x), z\right)\right) \\
& =\left(x, f_{q}(z, x)\right) \tag{1.13}
\end{align*}
$$

Since $f_{q}$ is smooth, $\varphi_{p} \circ \varphi_{q}^{-1}$ is a composition of smooth functions between Euclidean spaces, and hence $\varphi_{p} \circ \varphi_{q}^{-1}$ is smooth. This proves $\mathcal{A}$ is smooth, and thus $(S, \tau, \mathcal{A})$ is a smooth manifold of dimension 2. This concludes the proof.

Now that we have the structure to develop the theory of differentiability, we may turn our attention back to our original goal of obtaining the velocity of a ship from the charts. For now, we won't be able to compute the velocity. One must first know whether a function is differentiable before trying to differentiate it.

## Definition 20 [ ${ }^{\mathrm{k}}$ Maps]:

Let $\left(M, \tau_{M}, \mathcal{A}_{M}\right)$ and $\left(N, \tau_{N}, \mathcal{A}_{N}\right)$ be $\mathcal{C}^{k}$-manifolds with $\operatorname{dim} M=m$ and $\operatorname{dim} N=n$ and let $p \in M$. A map $f: M \rightarrow N$ is said to be of class $\mathcal{C}^{k}$ at $p$ if, and only if, there are charts $(\mathrm{U}, \varphi) \in \mathcal{A}_{M}$ and $(\mathrm{V}, \psi) \in \mathcal{A}_{N}$ with $p \in \mathrm{U}$ and $\mathrm{f}(\mathrm{U}) \subseteq \mathrm{V}$ such that $\psi \circ \mathrm{f} \circ \varphi^{-1}$ is of class $\mathcal{C}^{k}$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$.

The map $f$ is said to be of class $\mathcal{C}^{k}$ if, and only if, it is of class $\mathcal{C}^{k}$ at $p$ for every $p \in M$. A $\mathcal{C}^{\infty}$ map is often called a smooth map or a differentiable map. The map $\psi \circ \mathrm{f} \circ \varphi^{-1}$ is said to be a local representation of $f$.

The definition of a $\mathcal{C}^{k}$ map can be visualized through the following diagram:


## Proposition 21:

The notion of a $\mathcal{C}^{\mathrm{k}}$ map between two $\mathcal{C}^{\mathrm{k}}$-manifolds is well-defined, id est, it does not depend on the charts chosen.

Proof:
Let $\left(M, \tau_{M}, \mathcal{A}_{M}\right)$ and $\left(N, \tau_{N}, \mathcal{A}_{N}\right)$ be $\mathcal{C}^{k}$-manifolds with $\operatorname{dim} M=m$ and $\operatorname{dim} N=n$ and let $p \in M$. Let $f: M \rightarrow N$ be a map and let there be are charts $(U, \varphi) \in \mathcal{A}_{M}$ and $(\mathrm{V}, \psi) \in \mathcal{A}_{N}$ with $p \in \mathrm{U}$ and $\mathrm{f}(\mathrm{U}) \subseteq \mathrm{V}$ such that $\psi \circ \mathrm{f} \circ \varphi^{-1}$ is of class $\mathcal{C}^{k}$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$. We want to show that if there are charts $(W, \zeta) \in \mathcal{A}_{M}$ and $(X, \xi) \in \mathcal{A}_{N}$ with $p \in W$ and $f(W) \subseteq X$ such that $\xi \circ f \circ \zeta^{-1}$ is of class $\mathcal{C}^{k}$ (in the sense of Real Analysis) at $\zeta^{-1}(p)$.

Notice that $p \in U \cap W$ and $f(U \cap W) \subseteq V \cap X$. We are thus invited to consider the diagram


The diagram then invites us to notice that

$$
\begin{equation*}
\xi \circ f \circ \zeta^{-1}=\left(\xi \circ \psi^{-1}\right) \circ\left(\psi \circ f \circ \varphi^{-1}\right) \circ\left(\varphi \circ \zeta^{-1}\right), \tag{1.14}
\end{equation*}
$$

which, due to the fact that $\mathcal{A}_{M}$ and $\mathcal{A}_{N}$ are $\mathcal{C}^{k}$-atlases, is a composition of $\mathcal{C}^{k}$-maps in the sense of Real Analysis. Hence, $\xi \circ f \circ \zeta^{-1}$ is $\mathcal{C}^{k}$ in $\zeta(U \cap W)$ and, in particular, in $\zeta(p)$, proving the result.

## Proposition 22:

Let $\left(\mathrm{M}, \tau_{M}, \mathcal{A}_{M}\right)$ and $\left(\mathrm{N}, \tau_{\mathrm{N}}, \mathcal{A}_{\mathrm{N}}\right)$ be $\mathcal{C}^{\mathrm{k}}$-manifolds and let $\mathrm{p} \in \mathrm{M}$. Let $\mathrm{f}: \mathrm{M} \rightarrow \mathrm{N}$ be a $\mathcal{C}^{\mathrm{k}}$-map at $\mathrm{p} \in \mathrm{M}$. f is a $\mathcal{C}^{\mathrm{l}}$-map at $\mathrm{p} \in \mathrm{M}$ for every $0 \leqslant l \leqslant \mathrm{k}$. In particular, f is continuous at $p \in M$.

Proof:
We know there be are charts $(\mathrm{U}, \varphi) \in \mathcal{A}_{M}$ and $(\mathrm{V}, \psi) \in \mathcal{A}_{N}$ with $\mathrm{p} \in \mathrm{U}$ and $\mathrm{f}(\mathrm{U}) \subseteq \mathrm{V}$ such that $\psi \circ \mathrm{f} \circ \varphi^{-1}$ is of class $\mathcal{C}^{k}$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$. We know $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^{l}$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$, for every $0 \leqslant l \leqslant k$.

In particular, we see that $\psi \circ f \circ \varphi^{-1}$ is continuous. Since $\psi$ and $\varphi$ are homeomorphisms, it follows that

$$
\begin{equation*}
f=\psi^{-1} \circ\left(\psi \circ f \circ \varphi^{-1}\right) \circ \varphi \tag{1.15}
\end{equation*}
$$

is continuous at $p$.

## Proposition 23:

Let $\left(\mathrm{L}, \tau_{\mathrm{L}}, \mathcal{A}_{\mathrm{L}}\right),\left(\mathrm{M}, \tau_{M}, \mathcal{A}_{M}\right)$, and $\left(\mathrm{N}, \tau_{\mathrm{N}}, \mathcal{A}_{\mathrm{N}}\right)$ be $\mathcal{C}^{\mathrm{k}}$-manifolds and let $\mathrm{p} \in \mathrm{L}$. Let $\mathrm{f}: \mathrm{L} \rightarrow$ $M$ and $\mathrm{g}: \mathrm{M} \rightarrow \mathrm{N}$ be $\mathcal{C}^{\mathrm{k}}$-maps at $\mathrm{p} \in \mathrm{L}$ and at $\mathrm{f}(\mathrm{p}) \in \mathrm{M}$, respectively. Then the map $\mathrm{g} \circ \mathrm{f}: \mathrm{L} \rightarrow \mathrm{N}$ is $\mathcal{C}^{k}$ at $p \in \mathrm{~L}$.

Proof:
We know there are charts $(\mathrm{U}, \varphi) \in \mathcal{A}_{\mathrm{L}}$ and $(\mathrm{V}, \Psi) \in \mathcal{A}_{M}$ such that $\mathrm{p} \in \mathrm{U}, \mathrm{f}(\mathrm{U}) \subseteq \mathrm{V}$ and $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^{k}$. Furthermore, there are charts $(W, \zeta) \in \mathcal{A}_{M}$ and $(X, \xi) \in \mathcal{A}_{N}$ with $f(p) \in W, g(W) \subseteq X$ and such that $\xi \circ g \circ \zeta^{-1}$ is $\mathcal{C}^{k}$. This can be represented through the diagrams


Fortunately, $f$ is continuous as per Proposition 22. Thus, $f^{-1}(V \cap W)$ is an open set. Since $f(U) \subseteq V$, we see that $f^{-1}(V \cap W) \subseteq U$. Thus, we may consider the chart $\left(f^{-1}(V \cap W), \varphi\right)$ and the following diagram:


We see we may write

$$
\begin{equation*}
\xi \circ(g \circ f) \circ \varphi^{-1}=\left(\xi \circ g \circ \zeta^{-1}\right) \circ\left(\zeta \circ \psi^{-1}\right) \circ\left(\psi \circ f \circ \varphi^{-1}\right), \tag{1.16}
\end{equation*}
$$

which is just a composition of $\mathcal{C}^{k}$ maps in the sense of Real Analysis. Hence, we have found charts $\left(f^{-1}(\mathrm{~V} \cap \mathrm{~W}), \varphi\right) \in \mathcal{A}_{\mathrm{L}}$ and $(\mathrm{X}, \xi) \in \mathcal{A}_{\mathrm{N}}$ with $p \in \mathrm{f}^{-1}(\mathrm{~V} \cap \mathrm{~W}),(\mathrm{g} \circ \mathrm{f})\left(\mathrm{f}^{-1}(\mathrm{~V} \cap \mathrm{~W})\right) \subseteq X$ and such that $\xi \circ(g \circ f) \circ \varphi^{-1}$ is $\mathcal{C}^{k}$ at $p \in L$. We may thus conclude $g \circ f$ is $\mathcal{C}^{k}$ at $p$.

## Notation:

We shall often be lazy and say "Let $M$ be a manifold" instead of "Let $(M, \tau, \mathcal{A})$ be a manifold" for simplicity. Whenever this happens, the topology and atlas of the manifold should be clear from context.

One should notice this is just depraved notation and the manifold is the triple, not simply the set.

## Notation:

Given two $\mathcal{C}^{k}$-manifolds $M, N$, we denote by $\mathcal{C}^{k}(M, N)$ the space of all $\mathcal{C}^{k}$ functions $\mathrm{f}: \mathrm{M} \rightarrow \mathrm{N}$.

We shall often write $\mathcal{C}^{k}(M) \equiv \mathcal{C}^{k}(M, \mathbb{R})$.

## Remark:

From now on, we shall focus on the theory of differentiable (id est, smooth) manifolds instead of $\mathcal{C}^{\mathrm{k}}$-manifolds.

## Definition 24 [Diffeomorphisms]:

Let $M$ and $N$ be smooth manifolds and let $f: M \rightarrow N$ be a function. $f$ is said to be a diffeomorphism if, and only if, it is invertible and both $f$ and $f^{-1}$ are differentiable. Under this condition, M and N are said to be diffeomorphic.
f is said to be a local diffeomorphism at a point $p \in M$ if, and only if, there are open neighbourhoods $U \subseteq M$ and $V \subseteq N$ with $p \in U$ and $f(p) \in V$ such that $\left.f\right|_{U}: U \rightarrow V$ is a homeomorphism, where $\left.f\right|_{U}(p)=f(p), \forall p \in U$.

## Theorem 25:

Let $M$ and $N$ be smooth manifolds. The relation $M \simeq N \Leftrightarrow M$ and $N$ are diffeomorphic is an equivalence relation.
Proof:
One can check that the identity map id : $M \rightarrow M$ that maps $p \mapsto p$ is a diffeomorphism, for given a chart $(U, \varphi)$, one has $\varphi \circ$ id $\circ \varphi^{-1}=\mathrm{id}_{\mathbb{R}^{n}}$, which is smooth. Thus, $M \simeq M$.

If $f: M \rightarrow N$ is a diffeomorphism (meaning $M \simeq N$ ), $f^{-1}$ is also a diffeomorphism and we see that $N \simeq M$.

Finally, suppose $L \simeq M$ and $M \simeq N$ with diffeomorphisms $f: L \rightarrow M$ and $g: M \rightarrow N$. Proposition 23 guarantees $g \circ f: L \rightarrow N$ is a diffeomorphism and thus $L \simeq N$.

Definition 26 [Support of a Function]:
Let $(X, \tau)$ be a topological space. Let $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{V}$, where V is a vector space. The support of $f$, denoted supp $f$, is defined through

$$
\begin{equation*}
\operatorname{supp} f=\overline{\{x \in X ; f(x) \neq 0\}} \tag{1.17}
\end{equation*}
$$

where 0 stands for the null vector.

## Lemma 27:

Let $p \in \mathbb{R}^{n}$ and $0<\delta<r$. There is a function $\beta \in \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$ taking values in $[0,1]$, $\beta\left(\mathcal{B}_{\delta}(p)\right)=\{1\}$ and with compact support in $\mathcal{B}_{r}(p)$.
Proof:
Let $\epsilon>0$ be such that $\delta<\epsilon<r$. Consider the function

$$
\begin{equation*}
\beta(x)=\frac{\int_{\|x\|}^{\epsilon} g(t) d t}{\int_{\delta}^{\epsilon} g(t) d t}, \tag{1.18}
\end{equation*}
$$

where

$$
g(t)= \begin{cases}e^{-(t-\delta)^{-1}} e^{(t-\epsilon)^{-1}} & \text { for } \delta<t<\epsilon  \tag{1.19}\\ 0 & \text { otherwise }\end{cases}
$$

It can be shown (for example by handling it as an exercise to a Calculus student) that $g(t)$ is smooth. Hence, so is $\beta(x)$.

Notice that for $\|x\| \geqslant \epsilon$ (id est, $\left.x \notin \mathcal{B}_{\epsilon}(0)\right)$, it holds that $\beta(x)=0$. Hence,

$$
\begin{equation*}
\operatorname{supp} \beta \subseteq \overline{\mathcal{B}_{\epsilon}(0)} \subset \mathcal{B}_{r}(0) \tag{1.20}
\end{equation*}
$$

supp $\beta$ is compact by the Heine-Borel Theorem.
The generic case of a ball centered at any point follows through composition with a translation.

Definition 28 [Diameter of a Subset of a Metric Space]:
Let $(M, d)$ be a metric space and $A \subseteq M$ be a bounded set, $i d$ est, let there be $r>0$ and $p \in M$ such that $A \subseteq \mathcal{B}_{r}(p)$. We define the diameter of $A$, denoted diam $A$, through

$$
\begin{equation*}
\operatorname{diam} A=\sup _{x, y \in \mathcal{A}} d(x, y) . \tag{1.21}
\end{equation*}
$$

## Lemma 29:

Let $\mathrm{K} \subseteq \mathbb{R}^{n}$ be a compact set and $\mathrm{O} \subseteq \mathbb{R}^{n}$ be an open set such that $\mathrm{K} \subseteq \mathrm{O}$. Then there is a function $\beta \in \mathcal{C}^{\infty}\left(\mathbb{R}^{\mathrm{n}}\right)$ taking values in $[0,1], \beta(\mathrm{K})=\{1\}$ and has compact support in O .
Proof:
For each $p \in K$, let $U_{p}$ be an open ball centered at $p$ such that $U_{p} \subseteq O$ and $K_{p}$ be the closure of the open ball centered at $p$ with half the radius of $U_{p}$. The Heine-Borel Theorem ensures every $\mathrm{K}_{\mathrm{p}}$ is compact.

Notice that the collection $\left\{\stackrel{\circ}{\mathrm{K}}_{\mathrm{p}}\right\}_{p \in K}$ is an open cover of $K$. Since $K$ is compact, there is a finite subcover $\left\{\stackrel{\circ}{\mathrm{K}}_{p_{\lambda}}\right\}_{\lambda \in \Lambda}$ of $K$. For each $\lambda \in \Lambda$, Lemma 27 ensures, $\forall \lambda \in \Lambda$, the existence of a function $\beta_{\lambda} \in \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$ that vanishes outside of $U_{\mathfrak{p}_{\lambda}}$, is constantly 1 throughout $K_{p_{\lambda}}$ and has compact support in $\mathrm{U}_{\mathfrak{p}_{\lambda}}$.

We may now define

$$
\begin{equation*}
\beta(x)=1-\prod_{\lambda \in \Lambda}\left(1-\beta_{\lambda}(x)\right) . \tag{1.22}
\end{equation*}
$$

This function is a composition of smooth functions (hence it is smooth), is constantly 1 throughout $K$ (for $K \subseteq \bigcup_{\lambda \in \Lambda} K_{p_{\lambda}}$ ).

Notice that $\beta(x)=0 \Leftrightarrow \beta_{\lambda}(x)=0, \forall \lambda \in \Lambda$. Thus, $\operatorname{supp} \beta=\bigcup_{\lambda} \operatorname{supp} \beta_{\lambda}$. Since $\operatorname{supp} \beta_{\lambda} \subseteq \mathrm{U}_{\mathrm{p}_{\lambda}}, \forall \lambda \in \Lambda$, and $\bigcup_{\lambda \in \Lambda} \mathrm{U}_{\mathrm{p}_{\lambda}} \subseteq \mathrm{O}$, it follows that supp $\beta \subseteq 0$.

Since K is compact, it is closed and bounded by the Heine-Borel Theorem. Let diam $\mathrm{K}=$ $d$ and $\epsilon$ be the supremum of the radii of the open balls $U_{p_{\lambda}}$. Notice that supp $\beta \subseteq \mathcal{B}_{d+\epsilon}(0)$, and hence $\operatorname{supp} \beta$ is bounded. Since it is already closed by definition, the Heine-Borel Theorem guarantees supp $\beta$ is compact.

Theorem 30 [Existence of Cut-Off Functions]:
Let M be a smooth manifold. Let $\mathrm{K} \subseteq \mathrm{M}$ be a compact set and $\mathrm{O} \subseteq \mathrm{M}$ be an open set such that $K \subseteq O$. Then there is a function $\beta \in \mathfrak{C}^{\infty}(M)$ taking values in $[0,1], \beta(K)=\{1\}$ and has compact support in O .
Proof:
Suppose firstly that there is a chart $(\mathrm{U}, \varphi)$ such that $\mathrm{K} \subseteq \mathrm{U}$. In this case, $\varphi(\mathrm{U})$ is an open set and $\varphi(\mathrm{K})$ is a compact set with $\varphi(\mathrm{K}) \subseteq \varphi(\mathrm{U})$. Lemma 29 ensures the existence of a function $\beta^{*} \in \mathfrak{C}^{\infty}\left(\mathbb{R}^{n}\right)$ taking values in $[0,1]$, with $\beta^{*}(\varphi(\mathrm{~K}))=\{1\}$ and with compact support in $\varphi(\mathrm{U})$. Therefore, $\beta=\beta^{*} \circ \varphi$ satisfies the requirements we have.

This notion can be illustrated in the following diagram.


Figure 1: Vectors in manifolds are more delicate than on Euclidean space: one can't add vectors defined on different points of the manifold


Suppose now K is not contained in the coordinate neighborhood of any chart. Since every atlas covers the whole manifold, we know the coordinate neighborhoods of the charts that compose the manifold's atlas cover $K$. For every chart $\left(U_{\lambda}, \varphi_{\lambda}\right)$ we can attribute compact sets $\mathrm{K}_{\lambda}$ with $\mathrm{K}_{\lambda} \subseteq \mathrm{U}_{\lambda}$.

Since $K$ is compact, we know there is a finite collection of charts $\left\{\left(U_{i}, \varphi_{i}\right)\right\}_{i \in I}$ such that $K \subseteq \bigcup_{i \in I} \stackrel{\circ}{K}_{i} \subseteq \bigcup_{i \in I} U_{i}$. We may, without any loss of generality, pick $U_{i} \subseteq O, \forall i \in I$, since $K \subseteq O$ and $\left(U_{i} \cap O,\left.\varphi_{i}\right|_{O}\right)$ is a chart just as good as $\left(U_{i}, \varphi_{i}\right)$.

For each $i \in I$ we are left with the case in which $K$ lies inside the coordinate neighborhood of a chart, and this yields a collection of functions $\beta_{i}$ as per the beginning of the proof. One might then notice that

$$
\begin{equation*}
\beta(x)=1-\prod_{i \in \mathrm{I}}\left(1-\beta_{i}(x)\right) \tag{1.23}
\end{equation*}
$$

satisfies the requires properties.

## 2 Tangent Spaces and Fiber Bundles

With a definition of differentiability at hands, we may once more search for the velocity of a ship navigating on Earth. However, once again I'll delay the subject and present a question: what do we mean by velocity?

As usual, we shall still think of velocity as the time derivative of space, which is a function of the real parameter we call time. In a more abstract manner, we can think of velocities as the derivatives of smooth curves defined on our manifold.

This seems simple enough, but there is an issue: while on Euclidean space one doesn't need to bother with the point in which the vector is defined. Figure 1 illustrates this with the fact that summing velocities defined on different points will yield us something that is not the velocity of any curve on the manifold (for it is not tangent to the manifold).

In order to understand how to define vectors on manifolds, let us begin by working with surfaces in Euclidean space (Proposition 19 guarantees these are manifolds if we are in 3D space) and then proceed to remove the unnecessary structure.

For a given point $p \in \mathbb{R}^{n}$, we define the geometric tangent space to $\mathbb{R}^{n}$ at $p$, denoted $\mathbb{R}_{\mathfrak{p}}^{n}$, as $\mathbb{R}_{\mathfrak{p}}^{n}=\left\{(p, v) ; v \in \mathbb{R}^{n}\right\}$. A geometric tangent vector to $\mathbb{R}^{n}$ at $p$ is then simply an element of $\mathbb{R}_{\mathfrak{p}}^{n}$. For simplicity, we write $\left.v_{\mathfrak{p}} \equiv v\right|_{p} \equiv(p, v) \in \mathbb{R}_{\mathfrak{p}}^{n}$.

Notice that $\mathbb{R}_{\mathfrak{p}}^{n}$ can be made into a vector space by introducing the operations $v_{p}+\mathfrak{u}_{p}=$ $(v+u)_{\mathfrak{p}}$ and $\lambda \cdot v_{\mathfrak{p}}=(\lambda \cdot v)_{p}$.

Given a surface $S \subseteq \mathbb{R}^{n}$, the tangent vectors to $S$ at a point $p \in S$ should then be simply a subset of $\mathbb{R}_{\mathrm{p}}^{n}$. The issue we face is we can't generalize this notion to arbitrary manifolds, since it highly depends on the ambient space. The structures we do have on manifolds are notions of functions, smoothness, coordinate charts, and so on. Thus, we should look for how tangent vectors relate to these concepts in Euclidean space.

When dealing with the theory of real-valued functions defined on $\mathbb{R}^{n}$, a concept that arises and is connected to the idea of a tangent vector is the notion of directional derivative. Indeed, given $v_{p} \in \mathbb{R}_{p}^{n}$, there is an operator $D_{v_{p}}: C^{\infty}(M) \rightarrow \mathbb{R}$ which associated a function with its directional derivative in the direction of $v$ at the point $p$. It is such that

$$
\begin{equation*}
D_{v_{p}} f=\left[\frac{d}{d t} f(p+t v)\right]_{t=0} \tag{2.1}
\end{equation*}
$$

As all good derivatives, these operators respect the Leibniz rule:

$$
\begin{align*}
D_{v_{p}}(f g) & =\left[\frac{d}{d t}(f(p+t v) g(p+t v))\right]_{t=0}^{\prime} \\
& =\left[g(p+t v) \frac{d}{d t} f(p+t v)+f(p+t v) \frac{d}{d t} g(p+t v)\right]_{t=0}, \\
& =g(p)\left[\frac{d}{d t} f(p+t v)\right]_{t=0}+f(p)\left[\frac{d}{d t} g(p+t v)\right]_{t=0}, \\
& =g(p) D_{v_{p}} f+f(p) D_{v_{p}} g . \tag{2.2}
\end{align*}
$$

In a similar fashion, linearity of $\frac{d}{d t}$ over $\mathbb{R}$ implies $D_{v_{p}}(f+g)=D_{v_{p}} f+D_{v_{p}} g$ and $D_{v_{p}}(\lambda \cdot f)=\lambda \cdot D_{v_{p}} f$.

Suppose now we have a basis $\left\{e_{i}\right\}_{i=1}^{n}$. We may write $v_{p}=\left.v^{i} e_{i}\right|_{\mathfrak{p}}$, with summation over repeated indices implied. The chain rule implies

$$
\begin{align*}
D_{v_{p}} f & =\left[\frac{d}{d t} f(p+t v)\right]_{t=0}, \\
& =\left.\frac{d}{d t}(p+t v)^{i} \frac{\partial}{\partial x^{i}} f\left(x_{1}, x_{2}, x_{3}\right)\right|_{\left(x_{1}, x_{2}, x_{3}\right)=p}, \\
& =v^{i} \frac{\partial}{\partial x^{i}} f(p) . \tag{2.3}
\end{align*}
$$

Motivated by these constructions, given a point $p \in \mathbb{R}^{n}$ we may define a derivation at $p$ as a $\mathbb{R}$-linear operator $w: \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right) \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
w(f g)=f(p) w g+g(p) w f \tag{2.4}
\end{equation*}
$$

This in principle seems to generalize $D_{v_{p}}$. We let $T_{p} \mathbb{R}^{n}$ denote the collection of all derivations at $p . T_{p} \mathbb{R}^{n}$ can be regarded as a linear space, for if $v$ and $w$ are derivations at $p$ and $\lambda \in \mathbb{R}$, then

$$
\begin{equation*}
(v+w)(f)=v(f)+w(f), \quad(\lambda \cdot v)(f)=\lambda \cdot v(f) \tag{2.5}
\end{equation*}
$$

can be shown to be $\mathbb{R}$-linear. Furthermore,

$$
\begin{align*}
(v+w)(f g) & =v(f g)+w(f g) \\
& =f(p) v g+g(p) v f+f(p) w g+g(p) w f \\
& =f(p)(v g+w g)+g(p)(v f+w f) \\
& =f(p)(v+w) g+g(p)(v+w) f \tag{2.6}
\end{align*}
$$

and thus $v+w$ is a derivation. A similar proof holds for $\lambda \cdot v$.
There are some more interesting properties about derivations. For instance, notice that if $v$ is a derivation at $p$, then

$$
\begin{align*}
v(1) & =v(1 \cdot 1) \\
& =1 \cdot v(1)+1 \cdot v(1) \\
& =2 v(1) \tag{2.7}
\end{align*}
$$

which implies $v(1)=0$. Linearity guarantees $v f=0$ for all constant functions $f$.
Furthermore, suppose $f(p)=g(p)=0$. Then of course

$$
\begin{align*}
v(f g) & =f(p) v g+g(p) v f \\
& =0+0 \\
& =0 . \tag{2.8}
\end{align*}
$$

Nevertheless, he truly remarkable result is the fact that derivations and tangent vectors are one and the same thing: the map $v_{p} \mapsto D_{v_{p}}$ is an isomorphism between $\mathbb{R}_{p}^{n}$ and $T_{p} \mathbb{R}^{n}$.

The fact that $v_{p} \mapsto D_{v_{p}}$ is linear can be seen from the decomposition of $D_{v_{p}}$ into a basis. Given $\lambda \in \mathbb{R}, v_{p}, w_{p} \in \mathbb{R}_{p}^{n}$ we have

$$
\begin{align*}
D_{v_{p}+\lambda w_{p}} f & =D_{(v+\lambda w)_{p}} f, \\
& =\left(v^{i}+\lambda w^{i}\right) \frac{\partial}{\partial x^{i}} f(p), \\
& =v^{i} \frac{\partial}{\partial x^{i}} f(p)+\lambda w^{i} \frac{\partial}{\partial x^{i}} f(p), \\
& =D_{v_{p}} f+\lambda D_{w_{p}} f, \tag{2.9}
\end{align*}
$$

$\forall f \in \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$.
In order to prove it is one-to-one, let us assume $D_{v_{p}}=0$, id est, $D_{v_{p}} f=0, \forall f \in \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$. The decomposition of $D_{v_{p}}$ in a basis shows this implies $v_{p}^{i}=0$ for all components of $v_{p}$, and hence $v_{p}$ has to be the null vector. Thus, the kernel of $v_{p} \mapsto D_{v_{p}}$ is the trivial subspace $\{0\}$ and we conclude the transformation is injective.

Finally, let $w \in T_{p} \mathbb{R}^{n}$. We want to prove there is some $v_{p} \in \mathbb{R}_{p}^{n}$ such that $D_{v_{p}}=w$.

Let $\left\{e_{i}\right\}_{i=1}^{n}$ be a basis for $\mathbb{R}^{n}$. Consider the functions $x^{j}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that $\chi^{j}\left(e_{i}\right)=\delta_{i}^{j}$, where $\delta_{i}^{j}$ stands for the Kronecker delta. Consider the tangent vector $v_{p}=\left.v^{i} e_{i}\right|_{p}$ where $v^{i}=w\left(x^{i}\right)$.

Given $\mathrm{f} \in \mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$, Taylor's Theorem* guarantees we may write (summation is implicit over repeated indices)

$$
\begin{equation*}
f(x)=f(p)+\frac{\partial f}{\partial x^{i}}\left(x^{i}-p^{i}\right)+\left(x^{i}-p^{i}\right)\left(x^{j}-p^{j}\right) \int_{0}^{1}(1-t) \frac{\partial^{2} f}{\partial x^{i} \partial x^{j}} d t . \tag{2.10}
\end{equation*}
$$

The last term is a product of functions that vanish at $x=p$. Thus, the last term vanishes under a derivation at $p$. Hence,

$$
\begin{align*}
w f & =w(f(p))+w\left(\frac{\partial f}{\partial x^{i}}\left(x^{i}-p^{i}\right)\right), \\
& =0+\frac{\partial f}{\partial x^{i}}\left(w\left(x^{i}\right)-w\left(p^{i}\right)\right), \\
& =\frac{\partial f}{\partial x^{i}} v^{i}, \\
& =v^{i} \frac{\partial}{\partial x^{i}} f(p), \\
& =D_{v_{p}} f . \tag{2.11}
\end{align*}
$$

Now we are in position to define what is a vector in an arbitrary manifold.

## Definition 31 [Tangent Space]:

Let $M$ be a smooth manifold and let $p \in M$. A derivation at $p$ is an $\mathbb{R}$-linear operator $\nu: \mathcal{C}^{\infty}(M) \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
v(\mathrm{fg})=\mathrm{f}(\mathrm{p}) v \mathrm{~g}+\mathrm{g}(\mathrm{p}) v \mathrm{f} \tag{2.12}
\end{equation*}
$$

The set of all derivations at a point $p \in M$ is denoted $T_{p} M$ and referred to as the tangent space to $M$ at $p$. An element of $T_{p} M$ is often called a tangent vector to $M$ at $p$.

## Lemma 32:

Let M be a smooth manifold, $\mathrm{p} \in \mathrm{M}$. Let $v \in \mathrm{~T}_{\mathrm{p}} \mathrm{M}$. The following hold:
i. if $\mathrm{f} \in \mathrm{C}^{\infty}(\mathrm{M})$ is a constant function, $v \mathrm{f}=0$;
ii. if $f, g \in \mathcal{C}^{\infty}(M)$ are such that $f(p)=g(p)=0$, then $v(f g)=0$.

Proof:
We begin by showing the result for $f(q)=1, \forall q \in M$.

$$
\begin{align*}
v(1) & =v(1 \cdot 1), \\
& =1 \cdot v(1)+1 \cdot v(1), \\
& =2 v(1), \tag{2.13}
\end{align*}
$$

[^2]which implies $v(1)=0$. Since $v$ is $\mathbb{R}$-linear, it follows that $v f=0$ for all constant functions f.

Suppose now $f(p)=g(p)=0$. It follows that

$$
\begin{align*}
v(\mathrm{fg}) & =\mathrm{f}(\mathrm{p}) v \mathrm{~g}+\mathrm{g}(\mathrm{p}) v \mathrm{f}, \\
& =0+0, \\
& =0 . \tag{2.14}
\end{align*}
$$

This concludes the proof.

## Proposition 33:

Let M be a smooth manifold and $\mathrm{p} \in \mathrm{M} . \mathrm{T}_{\mathrm{p}} \mathrm{M}$ is a real vector space when equipped with the operations $+: T_{p} M \times T_{p} M \rightarrow T_{p} M$ and $:: \mathbb{R} \times T_{p} M \rightarrow T_{p} M$ defined through

$$
\begin{equation*}
(v+w)(f)=v f+w f, \quad(\lambda \cdot v)(f)=\lambda \cdot v f \tag{2.15}
\end{equation*}
$$

for all $\mathrm{f} \in \mathrm{C}^{\infty}(\mathrm{M})$.
Proof:
Let us begin by proving $v+w \in T_{p} M, \forall v, w \in T_{p} M$. Linearity of $v+w$ is easily proven and we shall focus on showing $v+w$ is a derivation. Indeed, notice that

$$
\begin{align*}
(v+w)(\mathrm{fg}) & =v(\mathrm{fg})+w(\mathrm{fg}) \\
& =\mathrm{f}(\mathrm{p}) v g+\mathrm{g}(\mathrm{p}) v \mathrm{f}+\mathrm{f}(\mathrm{p}) w g+\mathrm{g}(\mathrm{p}) w \mathrm{f} \\
& =\mathrm{f}(\mathrm{p})(v \mathrm{~g}+w g)+\mathrm{g}(\mathrm{p})(v \mathrm{f}+w \mathrm{f}) \\
& =\mathrm{f}(\mathrm{p})(v+w) \mathrm{g}+\mathrm{g}(\mathrm{p})(v+w) \mathrm{f} \tag{2.16}
\end{align*}
$$

A similar argument applies to $\lambda \cdot v$.
The algebraic properties that characterize $T_{p} M$ as a vector space comes naturally from the fact that $v f$ is a real number for any $v \in T_{p} M$.

We shall prove that $T_{p} M$ is not only finite dimensional, but also has the same dimension as $M$ (even though $\operatorname{dim} T_{p} M$ should be understood in an algebraic sense and $\operatorname{dim} M$ in a topological sense). In order to do so, we will define a specific collection of derivations which shall be similar to partial derivatives. Afterwards, we shall prove that arbitrary derivations are just linear combinations of this particular set, just as directional derivatives in Real Analysis can be written in terms of the derivatives with respect to Cartesian coordinates.

## Notation:

Given a function $\mathrm{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}, \partial_{\mathrm{i}} \mathrm{f}$ denotes the partial derivative of f with respect to its $i$-th argument. For example,

$$
\begin{equation*}
\partial_{2} f(x, y, z)=\frac{\partial f}{\partial y} \tag{2.17}
\end{equation*}
$$

and so on.

Definition 34 [Partial Derivatives]:
Let $M$ be a smooth manifold, $p \in M$ and $(U, \varphi)$ be a chart such that $p \in U$. Let $f \in \mathcal{C}^{\infty}(M)$. We define a function $\frac{\partial f}{\partial \varphi^{i}}: U \rightarrow \mathbb{R}$ through

$$
\begin{equation*}
\frac{\partial f}{\partial \varphi^{i}}:=\partial_{i}\left(f \circ \varphi^{-1}\right)(\varphi(\mathfrak{p})) . \tag{2.18}
\end{equation*}
$$

Notice this definition could also be stated as

$$
\begin{equation*}
\frac{\partial f}{\partial \varphi^{i}}:=\lim _{h \rightarrow 0} \frac{\left(f \circ \varphi^{-1}\right)\left(\varphi^{1}(p), \ldots, \varphi^{i}(p)+h, \ldots, \varphi^{n}(p)\right)-\left(f \circ \varphi^{-1}\right)(\varphi(p))}{h} . \tag{2.19}
\end{equation*}
$$

Definition 35 [Partial Derivatives at a Point]:
Let $M$ be a smooth manifold, $p \in M$ and $(U, \varphi)$ be a chart such that $p \in U$. We define the operator $\left(\frac{\partial}{\partial \varphi^{i}}\right)_{p}: C^{\infty}(M) \rightarrow \mathbb{R}$ through

$$
\begin{equation*}
\left(\frac{\partial}{\partial \varphi^{i}}\right)_{p} f=\frac{\partial f}{\partial \varphi^{i}}(p) \tag{2.20}
\end{equation*}
$$

for every $f \in \mathcal{C}^{\infty}(M)$.

## Lemma 36:

Let M be a smooth manifold, $\mathrm{p} \in \mathrm{M}$ and $(\mathrm{U}, \varphi)$ be a chart such that $\mathrm{p} \in \mathrm{U}$. Then $\left(\frac{\partial}{\partial \varphi^{i}}\right)_{p} \in$ $T_{p} M$.
Proof:
Linearity of derivatives ensures linearity of $\left(\frac{\partial}{\partial \varphi^{i}}\right)_{p}$. We shall prove it is a derivation. Let $f, g \in \mathcal{C}^{\infty}(M)$.

$$
\begin{align*}
\left(\frac{\partial}{\partial \varphi^{i}}\right)_{p}(\mathrm{fg}) & =\frac{\partial(\mathrm{f} \cdot \mathrm{~g})}{\partial \varphi^{i}}, \\
& =\partial_{i}\left[(\mathrm{f} \cdot \mathrm{~g}) \circ \varphi^{-1}\right](\varphi(\mathfrak{p})), \\
& =\partial_{i}\left[\left(f \circ \varphi^{-1}\right) \cdot\left(\mathrm{g} \circ \varphi^{-1}\right)\right](\varphi(\mathfrak{p})), \\
& =\left[\left(\mathrm{g} \circ \varphi^{-1}\right)(\varphi(\mathfrak{p}))\right] \partial_{\mathfrak{i}}\left(\mathrm{f} \circ \varphi^{-1}\right)(\varphi(\mathfrak{p}))+\left[\left(\mathrm{f} \circ \varphi^{-1}\right)(\varphi(\mathfrak{p}))\right] \partial_{\mathfrak{i}}\left(\mathrm{g} \circ \varphi^{-1}\right)(\varphi(\mathfrak{p})), \\
& =\mathrm{g}(\mathfrak{p}) \frac{\partial \mathrm{f}}{\partial \varphi^{i}}+\mathrm{f}(\mathfrak{p}) \frac{\partial \mathrm{g}}{\partial \varphi^{i}}, \\
& =\mathrm{g}(\mathfrak{p})\left(\frac{\partial}{\partial \varphi^{i}}\right)_{\mathrm{p}} \mathrm{f}+\mathrm{f}(\mathfrak{p})\left(\frac{\partial}{\partial \varphi^{i}}\right)_{\mathfrak{p}} \mathrm{g} . \tag{2.21}
\end{align*}
$$

This concludes the proof.

## Lemma 37:

Let M be a smooth manifold and $\mathrm{p} \in \mathrm{M}$. Let $v \in \mathrm{~T}_{\mathrm{p}} \mathrm{M}$. The following results hold:
i. if $f, g \in \mathcal{C}^{\infty}(M)$ are equal on a neighborhood of $p$, then $v(f)=v(g)$;
ii. if $\mathrm{h} \in \mathrm{C}^{\infty}(\mathrm{M})$ is constant on a neighborhood of p , then $v(\mathrm{~h})=0$.

Proof:
i. Since $v$ is linear, we want to prove that $v(f-g)=0$ if $f=g$ on some neighborhood U of $p$. Hence, we want to prove that $v(f)=0$ whenever f vanishes on some neighborhood $U$ of $p$.

Consider a function $\beta \in \mathcal{C}^{\infty}(M)$ such that $\beta(p)=1$ and with $\beta(q)=0, \forall q \in U^{c}$. The existence of such a function is ensured by Theorem 30. Notice $f \cdot \beta=0$ (while f vanishes on $\mathrm{U}, \beta$ vanishes outside of it). Linearity of $v$ ensures $v(0)=0$. We thus have

$$
\begin{align*}
0 & =v(f \beta) \\
& =f(p) v \beta+\beta(p) v f \\
& =0 \cdot v \beta+1 \cdot v f \\
& =v f \tag{2.22}
\end{align*}
$$

ii. Linearity guarantees it suffices to prove the result for $h=1$ on a neighborhood $U$ of $p$, for any constant function $h$ can be written as $h=\alpha \cdot 1$ for $\alpha \in \mathbb{R}$. Furthermore, the first item implies we may assume $h(p)=1, \forall p \in M$. Any function constant throughout $U$ will be equal to $\alpha h$ on $U$ and the previous item will enforce the result.

We have

$$
\begin{align*}
v(1) & =v(1 \cdot 1) \\
& =1 v(1)+1 v(1) \\
& =2 v(1) \tag{2.23}
\end{align*}
$$

Since $v(1)=2 v(1)$, we conclude $v(1)=0$.

A remark should be made at this point: even though the tangent space seems to have a local behaviour - for the derivations select the specific point under considerations and derivatives lie on the tangent space -, its elements act on functions belonging to $\mathcal{C}^{\infty}(M)$, which is a global property. Wouldn't it be expected that derivations may also act on elements of $\mathcal{C}^{\infty}(\mathrm{U})$ for some open neighborhood $U$ of $p$ ?

Consider the map $\Phi: T_{p} U \rightarrow T_{p} M$ such that $\Phi(v): \mathcal{C}^{\infty}(M) \rightarrow \mathbb{R}$ is given by $\Phi(v)(f)=$ $v\left(\left.f\right|_{u}\right)$, where $\left.f\right|_{u}: U \rightarrow \mathbb{R}$ is the map given by $\left.f\right|_{u}(p)=f(p), \forall p \in U$. Since $v$ is a linear derivation, it holds that $\Phi(v)$ is as well, and thus $\Phi(v) \in T_{p} M . \Phi$ is also a linear transformation. Our goal is to prove $\Phi$ is an isomorphism (and therefore $T_{p} U$ and $T_{p} M$ are identical in a linear way).

Suppose $\Phi(v)=0$. This means $v\left(\left.f\right|_{u}\right)=0, \forall f \in \mathcal{C}^{\infty}(M)$. Let $g \in \mathcal{C}^{\infty}(U)$ and let $\beta \in \mathcal{C}^{\infty}(M)$ be a function such that supp $\beta \subseteq U$ and $\beta(K)=\{1\}$ for a compact set $K$ with $\stackrel{\circ}{\mathrm{K}} \neq \varnothing$. The existence of such a function is guaranteed by Theorem 30. The existence of such $K \subseteq U$ is ensured by the fact that $M$ is locally Euclidean. $\beta \mathrm{g}$ can be understood as a function defined on all of $M$ that coincides with $g$ on some neighborhood of $p$. Notice
now that due to Lemma 37 we now see that

$$
\begin{align*}
v(\mathrm{~g}) & =v\left([\beta \mathrm{~g}]_{\mathrm{u}}\right), \\
& =\Phi(v)(\beta \mathrm{g}), \\
& =0 . \tag{2.24}
\end{align*}
$$

Therefore, $\operatorname{Ker} \Phi$ is the trivial linear subspace, which proves $\Phi$ is one-to-one.
Let now $v \in T_{p} M$. We want to find $w \in T_{p} U$ such that $\Phi(w)=v$. Given $\beta$ as above, we may define $w(f)=v(\beta f), \forall f \in \mathcal{C}^{\infty}(\mathrm{U})$. This yields

$$
\begin{align*}
\Phi(w) f & =w\left(\left.\mathrm{f}\right|_{\mathrm{U}}\right), \\
& =v\left(\left.\beta \mathrm{f}\right|_{\mathrm{U}}\right), \\
& =v(\beta \mathrm{f}), \\
& =v(\mathrm{f}), \tag{2.25}
\end{align*}
$$

where we used the fact that $\beta f$ and $f$ coincide in a neighborhood of $p$.
Therefore, given any chart $(U, \varphi)$ with $p \in U$, it holds that $T_{p} U$ and $T_{p} M$ are isomorphic. In particular, given two charts $(\mathrm{U}, \varphi)$ and $(\mathrm{V}, \psi)$ with $p \in \mathrm{U} \cap \mathrm{V}$ it also holds that $T_{p} \mathrm{U}$ and $T_{p} V$ are isomorphic, since isomorphisms are an equivalence relation between linear spaces.

In particular, we may study $T_{p} M$ by choosing a particular chart $(U, \varphi)$ such that $x, y \in \varphi(\mathrm{U}) \Rightarrow \mathrm{tx}+(1-\mathrm{t}) \mathrm{y} \in \varphi(\mathrm{U}), \forall \mathrm{t} \in[0,1]$. The existence of such charts in ensured by the fact that $\mathbb{R}^{n}$ is a locally convex space. Given an arbitrary chart $(V, \psi)$, we know there is a convex open set $\mathrm{O} \subseteq \psi(\mathrm{V})$. Since $\psi$ is a homeomorphism, $\mathrm{U}=\psi^{-1}(\mathrm{O})$ is open and we may define $\varphi=\left.\psi\right|_{\mathrm{u}}$. For more information, see [1].

## Theorem 38:

Let M be a smooth manifold with $\operatorname{dim} \mathrm{M}=\mathrm{n}$ and let $(\mathrm{U}, \varphi)$ be a chart with $\mathrm{p} \in \mathrm{U}$. It holds that

$$
\begin{equation*}
\left(\left(\frac{\partial}{\partial \varphi^{1}}\right)_{p}, \ldots,\left(\frac{\partial}{\partial \varphi^{n}}\right)_{p}\right) \tag{2.2.2}
\end{equation*}
$$

is a basis for $\mathrm{T}_{\mathrm{p}} \mathrm{M}$. If $\varphi^{i}: M \rightarrow \mathrm{R}$ is the i -th coordinate function of $\varphi$, we may write, for any $v \in T_{p} M$,

$$
\begin{equation*}
v=v\left(\varphi^{i}\right)\left(\frac{\partial}{\partial \varphi^{n}}\right)_{p}^{\prime}, \tag{2.27}
\end{equation*}
$$

where once again summation is implied over repeated indices.
Proof:
The previous discussion allow us to pick, without any loss of generality, $(\mathrm{U}, \varphi)$ to be such that $\varphi(\mathrm{U})$ is convex. A translation allow us to choose $\varphi$ such that the chart is centered. A derivation always vanishes once applied to a constant, so the translation is meaningless from its point of view.

Let $\mathrm{g} \in \mathrm{C}^{\infty}(\varphi(\mathrm{U}))$. The Taylor Formula with remainder in integral form allows us to write, $\forall x \in \varphi(\mathrm{U})$,

$$
\begin{equation*}
g(x)=g(0)+\int_{0}^{1} \frac{\partial g}{\partial x^{i}} d t x^{i} \tag{2.28}
\end{equation*}
$$

where summation is implied over repeated indices. For simplicity, we define

$$
\begin{equation*}
\mathrm{g}_{\mathrm{i}}(\mathrm{x})=\int_{0}^{1} \frac{\partial \mathrm{~g}}{\partial \mathrm{x}^{\mathrm{i}}} \mathrm{dt} . \tag{2.29}
\end{equation*}
$$

Notice that $\mathrm{g}_{\mathrm{i}}(0)=\left(\frac{\partial \mathrm{g}}{\partial \mathrm{x}^{\mathrm{i}}}\right)_{0}$.
Let now $\mathrm{f} \in \mathcal{C}^{\infty}(M)$. We may define $\mathrm{g} \equiv \mathrm{f} \circ \varphi^{-1}$, according to the diagram below.


The Taylor formula for g now yields

$$
\begin{align*}
g(\varphi(q)) & =g(\varphi(p))+g_{i}(\varphi(q)) \varphi^{i}(q) \\
f(q) & =f(p)+f_{i}(q) \varphi^{i}(q) \tag{2.30}
\end{align*}
$$

for some functions $f_{i}$. We may now apply the derivations $\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p}$ to see

$$
\begin{align*}
\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} f & =\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} f(p)+\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p}\left(f_{i} \varphi^{i}\right), \\
& =0+f_{i}(p)\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} \varphi^{i}+\varphi^{i}(p)\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} f_{i}, \\
& =f_{i}(p)\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} \varphi^{i}, \\
& =f_{j}(p), \tag{2.31}
\end{align*}
$$

where we used the fact that $\left(\frac{\partial}{\partial \varphi^{\prime}}\right)_{p} \varphi^{i}=\delta_{\mathfrak{j}}^{i}$. This can be seen from the definition provided at Eq. (2.19).

Given $v \in T_{p} M$, we may apply it to an arbitrary function $f \in \mathcal{C}^{\infty}(M)$ and see that

$$
\begin{align*}
v f & =v f(p)+v\left(f_{i} \varphi^{i}\right), \\
& =0+v\left(\varphi^{i}\right) f_{i}(p)+\varphi^{i}(p) v\left(f_{\mathfrak{i}}\right), \\
& =v\left(\varphi^{\mathfrak{i}}\right) \mathrm{f}_{\mathfrak{i}}(p), \\
& =v\left(\varphi^{i}\right)\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p} \mathrm{f} . \tag{2.32}
\end{align*}
$$

Hence,

$$
\begin{equation*}
v=v\left(\varphi^{i}\right)\left(\frac{\partial}{\partial \varphi^{j}}\right)_{p}^{\prime} \tag{2.33}
\end{equation*}
$$

for all $v \in T_{p} M$. In particular, we see that $v=0$ has only null coefficients, which implies $\left(\left(\frac{\partial}{\partial \varphi^{1}}\right)_{p^{\prime}} \ldots,\left(\frac{\partial}{\partial \varphi^{n}}\right)_{p}\right)$ not only generates $T_{p} M$, but is also linearly independent. This proves $\left(\left(\frac{\partial}{\partial \varphi^{1}}\right)_{p^{\prime}}, \ldots,\left(\frac{\partial}{\partial \varphi^{n}}\right)_{p}\right)$ is a basis for $T_{p} M$.

## Corollary 39:

Let $M$ be a smooth manifold and $p \in M . \operatorname{dim} T_{p} M=\operatorname{dim} M$.
Proof:
Direct consequence of Theorem 38.
As interesting as the theory of tangent spaces can be, it only deals with vectors one point at a time. Physics commonly needs to deal with vector fields, which requires a way of viewing how a vector changes from a point to another in both space and time. Thus, we would like to have a structure connecting different $T_{p} M^{\prime}$ s in a consistent way. In order to do so, we shall refer to the theory of bundles.

## Definition 40 [Fiber Bundle]:

Let $M, F, E$ be topological spaces and $\pi: E \rightarrow M$ be a surjective continuous map. The quadruple ( $\mathrm{E}, \pi, \mathrm{M}, \mathrm{F}$ ) is said to be a fiber bundle over M with model fiber F if, and only if, $\forall p \in M$ there is a neighborhood $U$ of $p$ and a homeomorphism $\varphi: \pi^{-1}(U) \rightarrow U \times F$ such that the following diagram commutes

where $\pi_{1}: \mathrm{U} \times \mathrm{F} \rightarrow \mathrm{U}$ maps $(\mathrm{u}, \mathrm{f}) \mapsto \mathrm{u} . \varphi$ is said to be a local trivialization of E under $\mathrm{U} . \mathrm{E}$ is called the total space of the bundle, M is its base space, F its typical fiber and $\pi$ its projection. For each $p \in M, E_{p}=\pi^{-1}(p)$ is called the fiber over $p$.

If $M, F$, and $E$ are smooth manifolds, $\pi$ is smooth and the local trivializations can be taken to be diffeomorphisms, we say ( $\mathrm{E}, \pi, \mathrm{M}, \mathrm{F}$ ) is a smooth fiber bundle. If the fiber bundle admits a trivialization over the entire base (known as a global trivialization) $M$, it is said to be a trivial fiber bundle. If a smooth fiber bundle happens to admit a global trivialization which is a diffeomorphism, it is said to be smoothly trivial.

Example [Product Space is a Fiber Bundle]:
Let $M$ and $F$ be topological spaces and consider the product space $M \times F$. If we define $\pi: M \times F \rightarrow M$ such that $(m, f) \mapsto m$ - id est, if $\pi$ is the projection of $M \times F$ onto $M$ - then $(M \times F, \pi, M, F)$ is a fiber bundle.

Indeed, $\pi$ is continuous by the very definition of the product topology (the coarsest topology which maintains the projections continuous). It is surjective, for given any
$m \in M$, one might pick any $f \in F$ and have $\pi((m, f))=m$. Also, let $p \in M$ and let $U$ be a neighborhood of $p$. Notice that

$$
\begin{align*}
\pi^{-1}(\mathrm{U}) & =\{(\mathrm{m}, \mathrm{f}) \in \mathrm{M} \times \mathrm{F} ; \mathrm{m} \in \mathrm{U}\} \\
& =\mathrm{U} \times \mathrm{F} \tag{2.34}
\end{align*}
$$

Consider then the homeomorphism id: $\pi^{-1}(\mathrm{U}) \rightarrow \mathrm{U} \times \mathrm{F}$, where $\mathrm{id}(\mathrm{x})=x, \forall x \in \mathrm{U} \times \mathrm{F}$. The diagram

commutes, for $\pi=\pi \circ \mathrm{id}$.
Notice that since id can be defined throughout all of $M \times F,(M \times F, \pi, M, F)$ is a trivial fiber bundle.

This example might have given a hint on the idea behind fiber bundles: we want to deal with spaces that might not be product spaces, but that resemble product spaces locally.

Right now, we are particularly interested in a specific smooth fiber bundle: the tangent bundle of a manifold.

Definition 41 [Tangent Bundle]:
Let $M$ be a smooth manifold. We define the tangent bundle of $M$, denoted TM, through

$$
\begin{equation*}
T M=\left\{(p, v) ; p \in M, v \in T_{p} M\right\} . \tag{2.35}
\end{equation*}
$$

## Example [Tangent Bundle of $S^{1}$ ]:

We've seen that $S^{2}$ is a smooth manifold. A similar construction can be made to show that the unit circle, $S^{1}$, is a smooth manifold with $\operatorname{dim} S^{1}=1$.

For each point $p \in S^{1}$, we can picture the tangent space to $S^{1}$ at $p, T_{p} S^{1}$, as the tangent line to the circle at that point, given that $T_{p} S^{1}$ is a one-dimensional linear space. This is illustrated in Figure 2a

TS ${ }^{1}$ as a set can be thought of just as depicted on Figure 2a. However, that isn't what one usually means when speaking of the tangent bundle. We shall now see how it can be equipped naturally with a topology and a smooth structure and be regarded as a smooth manifold. By giving it such properties, we picture $\mathrm{TS}^{1}$ as in Figure 2b: a "smooth and non-intersecting union" of tangent spaces. No point is simultaneously on two tangent spaces and we will be able to move around the tangent bundle smoothly.

## Proposition 42:

Let $M$ be a smooth manifold of dimension $n$. TM can be regarded as a smooth manifold of dimension 2 n such that the map $\pi: \mathrm{TM} \rightarrow \mathrm{M}$ defined by $\pi((\mathrm{p}, v))=\mathrm{p}$ is smooth.

(a) Depiction of the collection of tangent spaces to $S^{1}$ at various points

(b) Depiction of the tangent bundle as a "smooth, non-intersecting union" of tangent spaces

Figure 2: Depiction of how the tangent bundle arises from a "smoothed" union of tangent spaces

Proof:
Our first step will be building what will become a smooth structure for TM. Afterwards we shall use it to obtain a topology.

Let $(U, \varphi)$ be a smooth chart on $M$. Notice that

$$
\begin{equation*}
\pi^{-1}(\mathrm{U})=\{(\mathrm{p}, v) \in \mathrm{TM} ; \mathrm{p} \in \mathrm{U}\} \tag{2.36}
\end{equation*}
$$

id est, $\pi^{-1}(\mathrm{U})$ is the collection of vectors* which are tangent to some $p \in U$.

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[^0]:    *With apologies to the Flat Earth Community

[^1]:    *The word "commutative" means following a path in the diagram yields the same result as following any other path with the same endpoints. These concepts turn out to be quite useful in areas such as Category Theory [6, 9]

[^2]:    *See, exempli gratia, [8] or Appendix C of [7]

[^3]:    *In fact, of ordered pairs $(p, v)$ with $p \in U$ and $v \in T_{p} M$, but we may think of it simply as the collection of vectors themselves

