# The High-Energy Cross Section for the $e^+e^- \to \mu^+\mu^-$ Process in QED

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In this work, elementary S-matrix theory is briefly explained and the expression for the differential cross-section of some process in terms of the S-matrix is obtained. With these in hand, the Feynman rules are used in order to compute the invariant amplitude for the $e^+e^- \to \mu^+\mu^-$ process in QED in the high-energy limit, which is then utilised in order to calculate the high-energy cross-section for the $e^+e^- \to \mu^+\mu^-$ process.

Keywords: quantum electrodynamics, $e^+e^- \to \mu^+\mu^-$ process, Feynman diagrams

## I. THEORETICAL PRELIMINARIES

### A. The Scattering Matrix and the Invariant Amplitude

Suppose we are performing some given scattering experiment with initial state $|\mathbf{k}_A \mathbf{k}_B\rangle$ and final state $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$. These are, by assumption, asymptotic states, meaning they are eigenstates of the non-interacting Hamiltonian of the system.

We want to calculate the probability that such an initial state will scatter in such a way that the final state will be the outcome of the interaction. In order to understand how to do so, let us make an analogy with Classical Mechanics (CM). In CM, these states would be uniform motion states, respecting an equation of the form $\mathbf{x}(t) = \mathbf{x_0} + \mathbf{v}t$. However, the interaction is described by an ODE system given by

$$\ddot{\mathbf{x}} = \frac{1}{m}\mathbf{F}(\mathbf{x}). \tag{1}$$

Given an incoming free state $|\phi\rangle$, there is a single interacting state $|\psi\rangle$ which coincides with $|\phi\rangle$ at $t \to -\infty$ - after all, the evolution of a state is deterministic, even in Quantum Mechanics (QM). In a similar manner, given an outgoing free state, there is a single interacting state which coincides with it for $t \to +\infty$.

In order to compute the overlap of the initial state $|\mathbf{k}_A \mathbf{k}_B\rangle$ and the final state $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$, we might simply obtain the interacting state associated to $|\mathbf{k}_A \mathbf{k}_B\rangle$ and then obtain the free outgoing state, $|\xi\rangle$, given by this interacting state. The overlap of the original states we were considering is then simply $\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \xi\rangle$.

There is a certain nuance in this idea. In terms of CM, we are assuming that even if the position behaves in a complicated way for t in a certain time interval, it behaves as a free state for $t \to -\infty$ and $t \to +\infty$. This isn't always true. For example, we could in principle collide a proton and an electron and have, as an outgoing state, the bound state of a hydrogen atom. Since we are not interested in the mathematical details nor in scattering theory, we shall leave these and other issues aside, specially considering the scattering process we want to compute is, indeed, simple enough.

In order to implement this notion mathematically into QM and Quantum Field Theory (QFT), we define the S-matrix (S stands for scattering), through (in the interaction picture)

$$\lim_{T \to +\infty} \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | U(T, -T) | \mathbf{k}_A \mathbf{k}_B \rangle = \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_A \mathbf{k}_B \rangle, \tag{2}$$

where $U(t, t_0)$ is the operator such that $|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$.

In a free theory, we get that $S = \mathbb{1}$. This motivates us to define the T-matrix (T stands for transfer) through $S = \mathbb{1} + iT$.

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Every scattering process must still conserve 4-momentum, and as a consequence this must be reflected in the S and T matrices, which should contain a term $\delta^{(4)}\left(k_A + k_B - \sum_f p_f\right)$. We might then factor this term out of the T-matrix and define the invariant amplitude, M, by

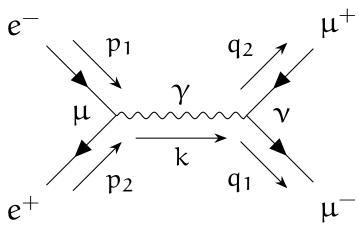
$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \cdot iM(k_A, k_B \to p_f). \tag{3}$$

We state, without proof, that the perturbative expansion of the invariant amplitude can be computed through[1]

$$iM = \text{sum of all connected, amputated Feynman diagrams.} \tag{4}$$

The coupling constant of Quantum Electrodynamics (QED) is the fine-structure constant, $\alpha \approx \frac{1}{137}$. Due to the fact that $\alpha \ll 1$, the first order approximation in perturbation theory provides a good approximation, since the second order is of order $\alpha^2 \approx \frac{1}{20\,000}$.

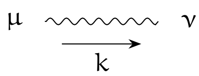
For the $e^+e^- \to \mu^+\mu^-$ process in QED, the first order approximation in perturbation theory contains a single diagram, which is




. (5)

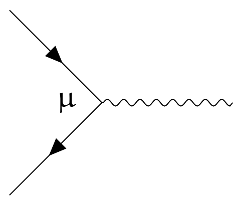
The value of each diagram can be computed via the Feynman Rules. For the vertices and legs appearing in diagram (5), these rules are

**Photon propagator:**




$$= \frac{-ig_{\mu\nu}}{k^2 + i\epsilon}; \tag{6}$$

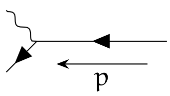
**QED vertex:**



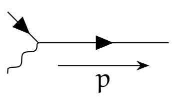

$$= iQe\gamma^{\mu}, \tag{7}$$

where $Q = -1$ for electrons;

**External fermions:**

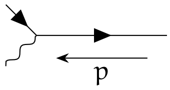



$$= u^s(p) \text{ (initial)}; \tag{8}$$

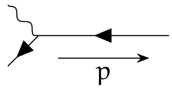



$$= \bar{u}^s(p) \text{ (final)}; \tag{9}$$

**External antifermions:**



$$= \bar{v}^s(p) \text{ (initial)}; \tag{10}$$



$$= v^s(p) \text{ (final)}. \tag{11}$$

### B. Cross Sections

Experimentally, the ways we can test QFT is through scattering experiments (hence the interest in defining an scattering matrix), and if the theory is falsifiable we must be able to compute some quantity which can be experimentally measured. One possible quantity is the cross section of a given process.

When colliding beams of particles, some particles will collide and bounce off each other while other particles won't. The cross section $\sigma$ is the fraction of time they do bounce off each other. For a flux $F$, which represents the number of incoming particles per unit time, the number of scattering events per unit time $N$ is, by definition of $\sigma$, given by

$$N = \sigma F. \tag{12}$$

The angular distribution of scattering events is not uniform. For an example, in Rutherford scattering we know there are more scattering events for low angles than there are for large angles. We might, thus, be interested in calculating the scattering probability as a function of the angles $\theta$ and $\phi$ of one of the final particles. Thus, we define the differential cross section $\frac{d\sigma}{d\Omega}$ as the differential probability per unit time per unit flux for a scattering event to happen in the solid angle $(\theta, \phi)$.

In general, in an scattering experiment, we are actually interested in the particles produced and their momenta. This motivates us to consider the differential cross section in terms of momenta, $\frac{d\sigma}{d^3p_1\cdots d^3p_n}$. If we integrate this quantity over some region of the momentum space, we recover the cross section for scattering into that region of the momentum space. In this case, we define the differential cross section $\frac{d\sigma}{d^3p_1\cdots d^3p_n}$ as the differential probability per unit time per unit flux for a scattering event to happen in the region around $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ in the momentum space.

Due to 4-momentum conservation, the components of final momenta are not independent, and four components must always be constrained. If there are only two outgoing particles, we are then left with two unconstrained components, which are usually taken to be the angles $\theta$ and $\phi$ of one of the particles, so that integration of $\frac{d\sigma}{d^3p_1\cdots d^3p_n}$ over the constrained momenta yields $\frac{d\sigma}{d\Omega}$ back.

Now we face the problem of obtaining the differential probability in terms of final momenta. We know that the differential probability, $dP$, of an initial state $|\mathbf{k}_A\mathbf{k}_B\rangle$ scatter into the final state $|\mathbf{p}_1\mathbf{p}_2\cdots\rangle$ must be given by

$$dP = \frac{|\langle \mathbf{p}_1\mathbf{p}_2\cdots|S|\mathbf{k}_A\mathbf{k}_B\rangle|^2}{\langle \mathbf{p}_1\mathbf{p}_2\cdots|\mathbf{p}_1\mathbf{p}_2\cdots\rangle\langle \mathbf{k}_A\mathbf{k}_B|\mathbf{k}_A\mathbf{k}_B\rangle}\, d\Pi\,, \tag{13}$$

where

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}\, d^3p_j\,. \tag{14}$$

In the last equation $V \equiv (2\pi)^3\delta^{(3)}(\mathbf{0})$ stands for the spatial volume and

$$E_\mathbf{p} \equiv \sqrt{\|\mathbf{p}\|^2 + m^2}. \tag{15}$$

We shall do this computation for a scalar field, but the expression we are going to obtain for the differential cross section holds for a spinor field as well.

Given two one-particle states $|\mathbf{p}\rangle$ and $|\mathbf{k}\rangle$, their relativistic normalization is such that[3]:

$$\langle \mathbf{p}|\mathbf{k}\rangle = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)} (\mathbf{p}-\mathbf{k}) . \tag{16}$$

Therefore,

$$\begin{aligned} \langle \mathbf{k}|\mathbf{k}\rangle &= (2\pi)^3 2E_{\mathbf{k}} \delta^{(3)} (\mathbf{0}) , \\ &= 2E_{\mathbf{k}} V. \end{aligned} \tag{17}$$

In a similar manner, we get

$$\langle \mathbf{k}_A \mathbf{k}_B | \mathbf{k}_A \mathbf{k}_B \rangle = 4E_{\mathbf{k}_A} E_{\mathbf{k}_B} V^2 \tag{18}$$

and

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \mathbf{p}_1 \mathbf{p}_2 \cdots \rangle = \prod_i 2E_{\mathbf{p}_i} V. \tag{19}$$

Furthermore, if we assume $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle \neq |\mathbf{k}_A \mathbf{k}_B\rangle$,

$$\begin{aligned} |\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_A \mathbf{k}_B \rangle|^2 &= |\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \mathbb{1} | \mathbf{k}_A \mathbf{k}_B \rangle + \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_A \mathbf{k}_B \rangle|^2 , \\ &= \left| (2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \cdot iM \right|^2 , \\ &= (2\pi)^8 \delta^{(4)} (0) \, \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \cdot |M|^2 , \\ &= (2\pi)^4 VT \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \cdot |M|^2 , \end{aligned} \tag{20}$$

where T is the total time available (VT amounts for the four-dimensional volume of space-time).

This yields

$$\begin{aligned} dP &= \frac{(2\pi)^4 VT |M|^2}{4E_{\mathbf{k}_A} E_{\mathbf{k}_B} V^2 \prod_i 2E_{\mathbf{p}_i} V} \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \prod_i \frac{V}{(2\pi)^3} d^3 p_i , \\ &= \frac{(2\pi)^4 T |M|^2}{4E_{\mathbf{k}_A} E_{\mathbf{k}_B} V} \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \prod_i \frac{V}{(2\pi)^3 \cdot 2E_{\mathbf{p}_i} V} d^3 p_i , \\ &= \frac{T}{V} \frac{(2\pi)^4 |M|^2}{4E_{\mathbf{k}_A} E_{\mathbf{k}_B}} \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \prod_i \frac{d^3 p_i}{(2\pi)^3 \cdot 2E_{\mathbf{p}_i}} . \end{aligned} \tag{21}$$

The differential cross section is the differential probability per unit time (hence we shall divide the above expression by T) per unit flux. In the rest frame of one of the particles, say A, the flux is given by $F = \frac{\|\mathbf{v}_B\|}{V}$. In an arbitrary frame of reference, $F = \frac{\|\mathbf{v}_A - \mathbf{v}_B\|}{V}$. If we divide dP by the time T and by the flux, we get the differential cross section:

$$\begin{aligned} d\sigma &= \frac{1}{T} \frac{V}{\|\mathbf{v}_A - \mathbf{v}_B\|} \frac{T}{V} \frac{(2\pi)^4 |M|^2}{4E_{\mathbf{k}_A} E_{\mathbf{k}_B}} \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \prod_i \frac{d^3 p_i}{(2\pi)^3 \cdot 2E_{\mathbf{p}_i}} , \\ &= \frac{(2\pi)^4 |M|^2}{4E_{\mathbf{k}_A} E_{\mathbf{k}_B} \|\mathbf{v}_A - \mathbf{v}_B\|} \delta^{(4)} \left( k_A + k_B - \sum_f p_f \right) \prod_i \frac{d^3 p_i}{(2\pi)^3 \cdot 2E_{\mathbf{p}_i}} . \end{aligned} \tag{22}$$

Now that the T and V terms have dropped out, we might take $V \to +\infty$ and $T \to +\infty$.

Since we are particularly interested in the $e^+e^- \to \mu^+\mu^-$ process, we might consider the special case in which there are only two outgoing particles. Furthermore, since the muon mass is much larger than the electron's, we might assume the electron mass is zero (and the same goes for the positron). Let us consider this situation in the center of mass reference frame.

In the center of mass, we get that $E_{\mathbf{k}_A} = E_{\mathbf{k}_B} \equiv \frac{1}{2}E_{CM}$ and $\mathbf{k}_A = -\mathbf{k}_B \equiv \mathbf{k}$. Since we are treating the electron as a massless particle, notice that $E_{CM} = \|\mathbf{k}\|$. Thus, we might write

$$\begin{aligned} d\sigma &= \frac{(2\pi)^4|M|^2}{4\frac{E_{CM}}{2}\frac{E_{CM}}{2}\left\|2\frac{\mathbf{k}}{E_{CM}} + 2\frac{\mathbf{k}}{E_{CM}}\right\|}\delta^{(4)}\left(2k - p_1 - p_2\right)\frac{d^3p_1\, d^3p_2}{(2\pi)^6 \cdot 2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}}, \\ &= \frac{|M|^2}{4E_{CM}^2}(2\pi)\delta^{(4)}\left(2k - p_1 - p_2\right)\frac{d^3p_1\, d^3p_2}{(2\pi)^3 \cdot 2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}}. \end{aligned} \tag{23}$$

If we want to obtain the actual cross section, or the differential cross section in terms of scattering angles, we must integrate $d\sigma$ over the momentum phase space. Let us first integrate over the three components of $\mathbf{p}_2$ and impose energy-momentum conservation through the delta functions available. Since we are in the center of mass, this shall only impose that $\mathbf{p}_1 = -\mathbf{p}_2$. Notice that since we are only considering the spatial integrals, this has not imposed the energy conservation constraints on $E_{\mathbf{p}_1}$ and $E_{\mathbf{p}_2}$ yet: each particle does respect the relativistic energy-momentum relation separately, but there is still no overall energy conservation on the scattering process. Since they obey the relativistic energy-momentum relation and $\mathbf{p}_1 = -\mathbf{p}_2$, we know that $E_{\mathbf{p}_i} = \sqrt{\|\mathbf{p}_1\|^2 + m_i^2}$. Since we are interested in the high-energy limit of the $e^+e^- \to \mu^+\mu^-$ process, we can take the muon and antimuon masses to be equal to zero.

We get for now that

$$\begin{aligned} \int d\sigma &= \frac{|M|^2}{4E_{CM}^2}\int (2\pi)\delta\left(E_{CM} - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}\right)\frac{d^3p_1}{(2\pi)^3 \cdot 2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}}, \\ &= \frac{|M|^2}{4E_{CM}^2}\int (2\pi)\delta\left(E_{CM} - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}\right)\frac{\|\mathbf{p}_1\|^2\, dp_1\, d\Omega}{(2\pi)^3 \cdot 2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}}, \\ &= \frac{|M|^2}{4E_{CM}^2}\int \delta\left(E_{CM} - 2\|\mathbf{p}_1\|\right)\frac{\|\mathbf{p}_1\|^2\, dp_1\, d\Omega}{16\pi^2 \cdot \|\mathbf{p}_1\|^2}, \\ &= \frac{|M|^2}{4E_{CM}^2}\int \frac{d\Omega}{16\pi^2}. \end{aligned} \tag{24}$$

Therefore, we get that, at the center of mass reference frame,

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{|M|^2}{64\pi^2 E_{CM}^2}. \tag{25}$$

In order to make notation simpler, we shall adopt the Mandelstam variable $s = (k_A + k_B)^2 = (p_1 + p_2)^2$ and write $s \equiv E_{CM}^2$ from now on. In this notation, we get that

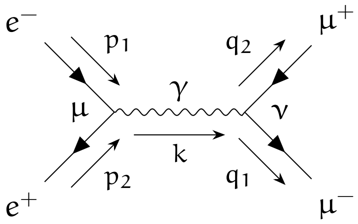
$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{|M|^2}{64\pi^2 s}. \tag{26}$$

## II. COMPUTATION OF THE INVARIANT AMPLITUDE

Eq. (26) shows that everything we must do now is compute the invariant amplitude $M$ and we are essentially done. As stated in Subsection I A, QED allows us to do perturbation theory due to the smallness of its coupling constant and therefore we might simply consider diagram (5) in the computation. The value of such a diagram can be obtained through the Feynman rules.

In fact, there is one more complication. Experimentally, the beams we collide are not polarized, which forces us to average $|M|^2$ over incoming spins. Furthermore, the muon detectors can't measure the muon and antimuon polarizations, and therefore we must sum $|M|^2$ over outgoing spins. In this way, our computation can actually be used in order to analyse experimentaldata. However, we shall not worry with this issue for now.

Applying the Feynman rules to diagram (5), we get




$$= \bar{v}(p_2)(-ie\gamma_\mu)u(p_1)\left(\frac{-ig^{\mu\nu}}{k^2+i\epsilon}\right)\bar{u}(q_1)(-ie\gamma_\nu)v(q_2), \tag{27}$$

*i.e.*,

$$\begin{aligned} iM &= \bar{v}(p_2)(-ie\gamma_\mu)u(p_1)\left(\frac{-ig^{\mu\nu}}{k^2+i\epsilon}\right)\bar{u}(q_1)(-ie\gamma_\nu)v(q_2), \\ &= \frac{ie^2}{k^2}\bar{v}(p_2)\gamma_\mu u(p_1)g^{\mu\nu}\bar{u}(q_1)\gamma_\nu v(q_2), \\ &= \frac{ie^2}{k^2}\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(q_1)\gamma^\mu v(q_2). \end{aligned} \tag{28}$$

In spinorial notation, we get

$$M = \frac{e^2}{k^2}\cdot[\bar{v}(p_2)]_{1a}\left[\gamma_\mu\right]_{ab}[u(p_1)]_{b1}\cdot[\bar{u}(q_1)]_{1c}[\gamma^\mu]_{cd}[v(q_2)]_{d1}\,. \tag{29}$$

The quantity we are interested in is $|M|^2$. In order to compute it, we will need to square two quantities that behave as $J^\mu = \bar{\psi}_1\gamma^\mu\psi_2$. Recalling that $\bar{\psi} = \psi^\dagger\gamma^0$ and $\gamma^0\gamma^0 = \mathbb{1}$, notice that

$$\begin{aligned} J^{\mu\dagger} &= \left[\bar{\psi}_1\gamma^\mu\psi_2\right]^\dagger, \\ &= \left[\psi_1^\dagger\gamma^0\gamma^\mu\psi_2\right]^\dagger, \\ &= \psi_2^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}\psi_1, \\ &= \psi_2^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\psi_1, \\ &= \bar{\psi}_2\gamma^\mu\psi_1, \end{aligned} \tag{30}$$

where we used the fact that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, proved in Appendix A.

In terms of spinorial indices, we then get that

$$\begin{aligned} J^{\mu\dagger}J^\nu &= \left[\bar{\psi}_2\right]_{1a}[\gamma^\mu]_{ab}[\psi_1]_{b1}\cdot\left[\bar{\psi}_1\right]_{1c}[\gamma^\nu]_{cd}[\psi_2]_{d1}\,, \\ &= [\psi_2]_{d1}\left[\bar{\psi}_2\right]_{1a}[\gamma^\mu]_{ab}[\psi_1]_{b1}\left[\bar{\psi}_1\right]_{1c}[\gamma^\nu]_{cd}\,, \\ &= \left[\psi_2\bar{\psi}_2\right]_{da}[\gamma^\mu]_{ab}\left[\psi_1\bar{\psi}_1\right]_{bc}[\gamma^\nu]_{cd}\,, \\ &= \mathrm{Tr}\left[\left(\psi_2\bar{\psi}_2\right)(\gamma^\mu)\left(\psi_1\bar{\psi}_1\right)(\gamma^\nu)\right]. \end{aligned} \tag{31}$$

### A. Dealing With Spin Through Trace Technology

As mentioned earlier, we have to bother with the issue that the beams we are colliding are not polarized. In order to do so, we consider, instead of simply $|M|^2$ for a single spin configuration, the quantity

$$\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}|M(s_1,s_2\to r_1,r_2)|^2, \tag{32}$$

where $s_1$ denotes the spin of the incoming electron, $s_2$ the spin of the incoming positron, $r_1$ the spin of the outgoing muon and $r_2$ the spin of the outgoing antimuon.

We also must consider the fact that the detectors can't measure the muon spins, and thus what we measure experimentally is the combined effect of all possible muon spin configurations. To solve this issue, we consider the quantity

$$\begin{aligned}\overline{|M|^2} &= \frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}\sum_{r_1}\sum_{r_2}|M(s_1,s_2\to r_1,r_2)|^2,\\ &= \frac{1}{4}\sum_{\text{spins}}|M|^2.\end{aligned} \tag{33}$$

Thus, we see that

$$\begin{aligned}\overline{|M|^2} &= \frac{e^4}{4k^4}\sum_{\text{spins}}[\bar{v}_{s_2}(p_2)\gamma_\nu u_{s_1}(p_1)\bar{u}_{r_1}(q_1)\gamma^\nu v_{r_2}(q_2)]^\dagger\,\bar{v}_{s_2}(p_2)\gamma_\mu u_{s_1}(p_1)\bar{u}_{r_1}(q_1)\gamma^\mu v_{r_2}(q_2),\\ &= \frac{e^4}{4k^4}\sum_{\text{spins}}[\bar{u}_{r_1}(q_1)\gamma^\nu v_{r_2}(q_2)]^\dagger\,[\bar{v}_{s_2}(p_2)\gamma_\nu u_{s_1}(p_1)]^\dagger\,\bar{v}_{s_2}(p_2)\gamma_\mu u_{s_1}(p_1)\bar{u}_{r_1}(q_1)\gamma^\mu v_{r_2}(q_2),\\ &= \frac{e^4}{4k^4}\sum_{\text{spins}}\mathrm{Tr}\big[[u_{s_1}(p_1)\bar{u}_{s_1}(p_1)]\,\gamma_\nu\,[v_{s_2}(p_2)\bar{v}_{s_2}(p_2)]\,\gamma_\mu\big]\,[\bar{u}_{r_1}(q_1)\gamma^\nu v_{r_2}(q_2)]^\dagger\,\bar{u}_{r_1}(q_1)\gamma^\mu v_{r_2}(q_2),\\ &= \frac{e^4}{4k^4}\sum_{\text{spins}}\mathrm{Tr}\big[[u_{s_1}(p_1)\bar{u}_{s_1}(p_1)]\,\gamma_\nu\,[v_{s_2}(p_2)\bar{v}_{s_2}(p_2)]\,\gamma_\mu\big]\,\mathrm{Tr}[[v_{r_2}(q_2)\bar{v}_{r_2}(q_2)]\,\gamma^\nu\,[u_{r_1}(q_1)\bar{u}_{r_1}(q_1)]\,\gamma^\mu].\end{aligned} \tag{34}$$

This expression looks cumbersome, but it can be simplified. Since the trace of a sum is the sum of the traces, we might use the following completeness relations

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{35}$$

A proof for these relations is provided at Appendix A.

We now get

$$\overline{|M|^2} = \frac{e^4}{4k^4}\mathrm{Tr}\big[[\not{p}_1 + m_e]\,\gamma_\nu\,[\not{p}_2 - m_e]\,\gamma_\mu\big]\,\mathrm{Tr}\big[[\not{q}_2 - m_\mu]\,\gamma^\nu\,[\not{q}_1 + m_\mu]\,\gamma^\mu\big]. \tag{36}$$

Since we are interested in the high-enery limit of the $e^+e^- \to \mu^+\mu^-$ process, we might simply take $m_e = m_\mu \approx 0$ and have

$$\overline{|M|^2} = \frac{e^4}{4k^4}\mathrm{Tr}\big[\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\big]\,\mathrm{Tr}\big[\not{q}_2\gamma^\nu\not{q}_1\gamma^\mu\big]. \tag{37}$$

We now get

$$\begin{aligned}\mathrm{Tr}\big[\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\big] &= \mathrm{Tr}\Big[{p_1}^\alpha\gamma_\alpha\gamma_\nu {p_2}^\beta\gamma_\beta\gamma_\mu\Big],\\ &= {p_1}^\alpha {p_2}^\beta\,\mathrm{Tr}\big[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\big].\end{aligned} \tag{38}$$

The trace of the product of four Dirac matrices is given by the formula

$$\mathrm{Tr}\big(\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\big) = 4\left(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta} - g_{\alpha\beta}g_{\mu\nu}\right), \tag{39}$$

which is proven at Appendix A. Using this equation, it follows that

$$\begin{aligned}\mathrm{Tr}\big[\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\big] &= 4{p_1}^\alpha {p_2}^\beta\left(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta} - g_{\alpha\beta}g_{\mu\nu}\right),\\ &= 4\left(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - {p_1}^\alpha p_{2\alpha}g_{\mu\nu}\right).\end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}\left[\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\right]\mathrm{Tr}\left[\not{q}_2\gamma^\nu\not{q}_1\gamma^\mu\right] &= 16\left(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}-g_{\mu\nu}(p_1\cdot p_2)\right)\left({q_2}^\mu{q_1}^\nu+{q_2}^\nu{q_1}^\mu-g^{\mu\nu}(q_2\cdot q_1)\right),\\
&= 16\big((p_1\cdot q_2)(p_2\cdot q_1)+(p_1\cdot q_1)(p_2\cdot q_2)-(p_1\cdot p_2)(q_1\cdot q_2)\\
&\quad+(p_1\cdot q_1)(p_2\cdot q_2)+(p_1\cdot q_2)(p_2\cdot q_1)-(p_1\cdot p_2)(q_1\cdot q_2)\\
&\quad-(q_2\cdot q_1)(p_1\cdot p_2)-(q_2\cdot q_1)(p_1\cdot p_2)+4(p_1\cdot p_2)(q_2\cdot q_1)\big),\\
&= 32\left((p_1\cdot q_1)(p_2\cdot q_2)+(p_1\cdot q_2)(p_2\cdot q_1)\right).
\end{aligned} \tag{41}$$

Plugging this result in the expression for $\overline{|M|^2}$ we find that

$$\overline{|M|^2} = \frac{8e^4}{k^4}\left((p_1\cdot q_1)(p_2\cdot q_2)+(p_1\cdot q_2)(p_2\cdot q_1)\right). \tag{42}$$

This accounts for all the dynamics of the process. We have already dealt with all the trouble due to the interaction itself and now we must turn our attention to the relativistic kinematics.

Let us choose the center of mass reference frame in order to do the kinematic calculations. An appropriate choice of the $z$ axis allows us to write $p_1=(E,E\hat{\mathbf{z}}), p_2=(E,-E\hat{\mathbf{z}})$. The muon and antimuon need not to be on the $z$-axis after the collision, and thus in general there is an angle $\theta$ between $\hat{\mathbf{z}}$ and $\hat{\mathbf{q}}_1$. An appropriate choice of the $y$ axis allows us to write $q_1=(E,E\hat{\mathbf{q}}), q_2=(E,-E\hat{\mathbf{q}})$ where $\hat{\mathbf{q}}=\sin\theta\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}$.

We then have the following relations:

$$\begin{aligned}
(p_1\cdot p_2) &= (q_1\cdot q_2),\\
&= (E,-E\hat{\mathbf{q}})\cdot(E,-E\hat{\mathbf{q}}),\\
&= 2E^2.
\end{aligned} \tag{43}$$

$$\begin{aligned}
s &\equiv (p_1+p_2)^2,\\
&= k^2,\\
&= (q_1+q_2)^2,\\
&= q_1^2+2(q_1\cdot q_2)+q_2^2,\\
&= 4E^2.
\end{aligned} \tag{44}$$

$$\begin{aligned}
(p_1\cdot q_1) &= (p_2\cdot q_2),\\
&= (E,-E\hat{\mathbf{z}})\cdot(E,E\hat{\mathbf{q}}),\\
&= E^2-E^2\hat{\mathbf{z}}\cdot\hat{\mathbf{q}},\\
&= E^2(1-\cos\theta).
\end{aligned} \tag{45}$$

$$\begin{aligned}
(p_1\cdot q_2) &= (p_2\cdot q_1),\\
&= (E,E\hat{\mathbf{z}})\cdot(E,E\hat{\mathbf{q}}),\\
&= E^2+E^2\hat{\mathbf{z}}\cdot\hat{\mathbf{q}},\\
&= E^2(1+\cos\theta).
\end{aligned} \tag{46}$$

Notice that by $E$ we mean the energy in the center of mass reference frame.

We now see that

$$\begin{aligned}
\overline{|M|^2} &= \frac{e^4}{2E^4}\left(E^4(1-\cos\theta)^2+E^4(1+\cos\theta)^2\right),\\
&= \frac{e^4}{2}\left(1-2\cos\theta+\cos^2\theta+1+2\cos\theta+\cos^2\theta\right),\\
&= e^4\left(1+\cos^2\theta\right).
\end{aligned} \tag{47}$$

### B. Dealing With Spin Through the Helicity Structure

In the previous subsection, we computed $\overline{|\mathcal{M}|^2}$ by employing the completeness relations and trace technology. Alternatively, we might deal with each possible spin configuration and then proceed to average and sum over spins.

In the Weyl representation,

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ \sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_{-s} \\ -\sqrt{p \cdot \bar{\sigma}}\xi_{-s} \end{pmatrix}. \tag{48}$$

If we associate $\xi_s$ with the physical spin component along the $z$ axis and have momentum along the $z$ direction ($p = (E, p_z \hat{\mathbf{z}})$) we get

$$u_\uparrow(p) = \begin{pmatrix} \sqrt{E - p_z}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E + p_z}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}, \quad u_\downarrow(p) = \begin{pmatrix} \sqrt{E + p_z}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E - p_z}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}, \tag{49}$$

$$v_\uparrow(p) = \begin{pmatrix} \sqrt{E + p_z}\begin{pmatrix}0\\1\end{pmatrix} \\ -\sqrt{E - p_z}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}, \quad v_\downarrow(p) = \begin{pmatrix} -\sqrt{E - p_z}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E + p_z}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}. \tag{50}$$

In the high-energy limit, which is our interest, we have $E = p_z$, yielding

$$u_\uparrow(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad u_\downarrow(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v_\uparrow(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v_\downarrow(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{51}$$

If we define the chirality projectors through

$$L = \frac{1}{2}(\mathbb{1} - \gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \frac{1}{2}(\mathbb{1} + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{52}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, then we get in the high-energy limit that

$$\begin{aligned} &Ru_\uparrow = u_\uparrow, \quad Lu_\downarrow = u_\downarrow, \quad Lv_\uparrow = v_\uparrow, \quad Rv_\downarrow = v_\downarrow, \\ &\bar{u}_\uparrow L = \bar{u}_\uparrow, \quad \bar{u}_\downarrow R = \bar{u}_\downarrow, \quad \bar{v}_\uparrow R = \bar{v}_\uparrow, \quad \bar{v}_\downarrow L = \bar{v}_\downarrow, \end{aligned} \tag{53}$$

for chirality and helicity coincide for massless particles.

Since each of the four particles involved might have two possible configurations ($\uparrow$ or $\downarrow$), there are 16 possible configurations. However, the current $\bar{\psi}_2\gamma^\mu\psi_1$ vanishes in the high-energy limit for two particles with the same handedness. For example, we have

$$\begin{aligned} \bar{v}_\uparrow\gamma^\mu u_\uparrow &= \bar{v}_\uparrow R\gamma^\mu R u_\uparrow, \\ &= \bar{v}_\uparrow RL\gamma^\mu u_\uparrow, \\ &= 0, \end{aligned} \tag{54}$$

where we used the facts that $\gamma^\mu R = L\gamma^\mu$ (which can be proved through brute-force computation) and $RL = 0$.

On the other hand, opposite handednesses yield

$$\begin{aligned} \bar{v}_\uparrow\gamma^\mu u_\downarrow &= \bar{v}_\uparrow R\gamma^\mu L u_\downarrow, \\ &= \bar{v}_\uparrow R^2\gamma^\mu u_\downarrow, \\ &= \bar{v}_\uparrow R^2\gamma^\mu u_\downarrow, \\ &\neq 0. \end{aligned} \tag{55}$$

This condition leaves us with two configurations for the electron-positron pair (↑↓ and ↓↑) and two more for the muon-antimuon pair, for a total of four possibilities.

Let us consider firstly the case of a right-handed electron and a left-handed positron, with the electron moving in the positive direction of the *z* axis and the positron moving in the negative direction of the same axis (center of mass reference frame).

The general expressions for the Dirac spinors in the high-energy limit are

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \to \frac{\sqrt{2E}}{2}\begin{pmatrix} (1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_s \\ (1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_s \end{pmatrix}, \tag{56}$$

$$v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\xi_{-s} \end{pmatrix} \to \frac{\sqrt{2E}}{2}\begin{pmatrix} (1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-s} \\ -(1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-s} \end{pmatrix}, \tag{57}$$

which are due to the expressions

$$\sqrt{p\cdot\sigma} = \frac{(p\cdot\sigma + m)}{\sqrt{2(E+m)}}, \quad \sqrt{p\cdot\bar{\sigma}} = \frac{(p\cdot\bar{\sigma} + m)}{\sqrt{2(E+m)}}. \tag{58}$$

Right-handed *spinors* satisfy $(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi = +\xi$, while left-handed spinors satisfy $(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi = -\xi$. A particle's handedness coincides with the spinor's handedness, but the antiparticle's handedness is opposite to the spinor's. We shall then define

$$u_{+(-)} = u_{\uparrow(\downarrow)}, \quad v_{-(+)} = v_{\downarrow(\uparrow)}. \tag{59}$$

Notice that for both a right-handed and a left-handed positron it holds that $(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi = +\xi$, with $\xi = \begin{pmatrix}1\\0\end{pmatrix}$. We then have

$$u_+(p_1) = \frac{\sqrt{2E}}{2}\begin{pmatrix} (1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi \\ (1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi \end{pmatrix} = \frac{\sqrt{2E}}{2}\begin{pmatrix} 0\xi \\ 2\xi \end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{60}$$

As for the right-handed positron, we get

$$v_+(p_2) = \frac{\sqrt{2E}}{2}\begin{pmatrix} (1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi \\ -(1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi \end{pmatrix} = \frac{\sqrt{2E}}{2}\begin{pmatrix} 0\xi \\ -2\xi \end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}. \tag{61}$$

The quantity we are interested in is

$$|M|^2 = \frac{e^4}{k^4}\left[\bar{v}_-(p_2)\gamma^\nu u_+(p_1)\bar{u}_{r_1}(q_1)\gamma_\nu v_{r_2}(q_2)\right]^\dagger \bar{v}_-(p_2)\gamma^\mu u_+(p_1)\bar{u}_{r_1}(q_1)\gamma_\mu v_{r_2}(q_2). \tag{62}$$

We have

$$\begin{aligned}
\bar{v}_-(p_2)\gamma^\mu u_+(p_1) &= 2E\begin{pmatrix}0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & \\ 0 & \end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \\
&= 2E\begin{pmatrix}0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \\
&= 2E\begin{pmatrix}0 & -1\end{pmatrix}\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix}.
\end{aligned} \tag{63}$$

If we substitute each component of $\sigma^\mu$, we obtain the following four-vector

$$\bar{v}_-(p_2)\gamma^\mu u_+(p_1) = -2E \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \tag{64}$$

which can be interpreted as circular polarization in the direction of the $z$-axis.

This solves half of the problem. We still must deal with the muon spins. We shall assume the outgoing muon is right-handed, while the antimuon is left-handed. The muon is emitted at an angle $\theta$ with the $z$-axis, and thus we might obtain the one-form $\bar{u}_{r_1}(q_1)\gamma_\mu v_{r_2}(q_2)$ by simply considering the fact that

$$\left[\bar{u}_+(q_1)\gamma_\mu v_-(q_2)\right]^\dagger = \bar{v}_-(q_2)\gamma^\mu u_+(q_1) \tag{65}$$

and performing a rotation on the $xz$-plane of the result we obtained for $\bar{v}_-(p_2)\gamma^\mu u_+(p_1)$. We get

$$\begin{aligned}
\bar{u}_+(q_1)\gamma_\mu v_-(q_2) &= -2E \left[ \begin{pmatrix} 1 & & & \\ & \cos\theta & & \sin\theta \\ & & 1 & \\ & -\sin\theta & & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \right]^\dagger, \\
&= -2E \left[ \begin{pmatrix} 0 \\ \cos\theta \\ i \\ -\sin\theta \end{pmatrix} \right]^\dagger, \\
&= -2E \begin{pmatrix} 0 & \cos\theta & -i & -\sin\theta \end{pmatrix}.
\end{aligned} \tag{66}$$

We now get

$$\begin{aligned}
|M|^2 &= \frac{e^4}{k^4} \left[\bar{v}_-(p_2)\gamma_\nu u_+(p_1)\bar{u}_+(q_1)\gamma^\nu v_-(q_2)\right]^\dagger \bar{v}_-(p_2)\gamma_\mu u_+(p_1)\bar{u}_+(q_1)\gamma^\mu v_-(q_2), \\
&= \frac{e^4}{k^4} \left[ 4E^2 \begin{pmatrix} 0 & \cos\theta & -i & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \right]^\dagger \cdot 4E^2 \begin{pmatrix} 0 & \cos\theta & -i & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \\
&= e^4 \left(\cos\theta + 1\right)^\dagger \cdot \left(\cos\theta + 1\right), \\
&= e^4 \left(\cos\theta + 1\right)^2.
\end{aligned} \tag{67}$$

This is the result for right-handed electron and muon and left-handed positron and antimuon, which we shall denote as $|M_{+-\to+-}|^2$. The other three possibilities are obtainable in a similar way, and are

$$|M_{+-\to+-}|^2 = |M_{-+\to-+}|^2 = e^4 \left(1+\cos\theta\right)^2, \quad |M_{+-\to-+}|^2 = |M_{-+\to+-}|^2 = e^4 \left(1-\cos\theta\right)^2. \tag{68}$$

If we sum all possibilities and divide by 4 (in order to average over incoming spins), we obtain

$$\overline{|M|^2} = e^2(1+\cos^2\theta), \tag{69}$$

which agrees with Eq. (47).

## III. COMPUTATION OF THE TOTAL CROSS-SECTION

Let us now compute the cross-section for the $e^+e^- \to \mu^+\mu^-$ process. Eqs. (26) and (69) yield that, in the center of mass reference frame,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta). \tag{70}$$

Since $\alpha = \frac{e^2}{4\pi}$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta). \tag{71}$$

In order to compute the total cross-section, we must simply integrate over the solid angle. We get that

$$\begin{aligned}\sigma &= \frac{\alpha^2}{4s}\int_0^{2\pi}\int_0^{\pi}(1+\cos^2\theta)\sin\theta\, d\theta\, d\phi\,,\\ &= \frac{\pi\alpha^2}{2s}\int_0^{\pi}\sin\theta+\cos^2\theta\sin\theta\, d\theta\,,\\ &= \frac{\pi\alpha^2}{2s}\frac{8}{3},\\ &= \frac{4\pi\alpha^2}{3s},\end{aligned} \tag{72}$$

which is the result we were seeking.

## REFERENCES

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## Appendix A: Proofs of Useful Mathematical Formulae

### 1. Adjoint of Dirac Matrices

We want to show that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.
In the Dirac-Pauli representation, we have

$$\gamma^0 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}. \tag{A1}$$

We can see directly that $\gamma^{0\dagger} = \gamma^0 = \gamma^0\gamma^0\gamma^0$ (for $\gamma^0\gamma^0 = \mathbb{1}$). For $\gamma^i$, notice that

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^\dagger = \begin{pmatrix}A^\dagger & C^\dagger\\ B^\dagger & D^\dagger\end{pmatrix}. \tag{A2}$$

Therefore,

$$\begin{aligned}\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}^\dagger &= \begin{pmatrix}0 & -\sigma_i^\dagger\\ \sigma_i^\dagger & 0\end{pmatrix},\\ &= \begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix},\end{aligned} \tag{A3}$$

where we used the hermiticity of the Pauli matrices.
Notice now that

$$\begin{gathered}\begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix} = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix},\\ \gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0,\end{gathered} \tag{A4}$$

proving that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, as desired.

### 2. Completeness Relations

We want to show that

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{A5}$$

We know that, in the Weyl representation,

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\xi_{-s} \end{pmatrix}, \tag{A6}$$

where $\xi_s$ are two-component linearly independent spinors supposed to be normalized, so that $\xi_s^\dagger\xi_s = 1$ for each $s$, and usually taken to be orthogonal. We shall make this assumptions, so that $\sum_s \xi_s\xi_s^\dagger = \mathbb{1}$.

Furthermore, $\sigma^\mu = (\sigma^0, \sigma^i)$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$, where $\sigma^0) = \mathbb{1}$ and $\sigma^i$ are the Pauli matrices.

Notice now that

$$\begin{aligned}
\sum_s u_s(p)\bar{u}_s(p) &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \begin{pmatrix} \xi_s^\dagger\sqrt{p\cdot\sigma} & \xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \\
&= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \begin{pmatrix} \xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix}, \\
&= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s\xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\xi_s\xi_s^\dagger\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\xi_s\xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\xi_s\xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} \end{pmatrix}.
\end{aligned} \tag{A7}$$

Notice now that

$$\begin{aligned}
(p\cdot\sigma)(p\cdot\bar{\sigma}) &= \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \begin{pmatrix} p_0 + p_3 & +p_1 - ip_2 \\ +p_1 + ip_2 & p_0 - p_3 \end{pmatrix}, \\
&= \begin{pmatrix} p_0^2 - p_1^2 - p_2^2 - p_3^2 & 0 \\ 0 & p_0^2 - p_1^2 - p_2^2 - p_3^2 \end{pmatrix}, \\
&= p^2\mathbb{1}, \\
&= m^2\mathbb{1}.
\end{aligned} \tag{A8}$$

Furthermore, since $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ in the Weyl representation, we see that

$$\begin{aligned}
\not{p} &= \gamma\cdot p, \\
&= \begin{pmatrix} 0 & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & 0 \end{pmatrix}, \\
&= \begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar{\sigma} & 0 \end{pmatrix}.
\end{aligned} \tag{A9}$$

Thus, we might conclude that

$$\begin{aligned}
\sum_s u_s(p)\bar{u}_s(p) &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}, \\
&= \not{p} + m.
\end{aligned} \tag{A10}$$

A similar calculation provides that

$$\sum_s v_s(p)\bar{v}_s(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix},$$
$$= \not{p} - m. \tag{A11}$$

### 3. Trace Technology

We want to show that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\rho) = 4\,(g^{\mu\nu}g^{\tau\rho} + g^{\mu\rho}g^{\nu\tau} - g^{\mu\tau}g^{\nu\rho})$.

Firstly, let us prove that $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$. Recall that the Dirac matrices respect the Clifford algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and notice that

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= \mathrm{Tr}(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu), \\
&= \mathrm{Tr}(2g^{\mu\nu}\mathbb{1}) - \mathrm{Tr}(\gamma^\nu\gamma^\mu), \\
&= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu), \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu},
\end{aligned} \tag{A12}$$

where we used the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\rho) &= \mathrm{Tr}(2g^{\mu\nu}\gamma^\tau\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\tau\gamma^\rho), \\
&= \mathrm{Tr}(2g^{\mu\nu}\gamma^\tau\gamma^\rho - 2\gamma^\nu g^{\mu\tau}\gamma^\rho + \gamma^\nu\gamma^\tau\gamma^\mu\gamma^\rho), \\
&= \mathrm{Tr}(2g^{\mu\nu}\gamma^\tau\gamma^\rho - 2\gamma^\nu g^{\mu\tau}\gamma^\rho + 2\gamma^\nu\gamma^\tau g^{\mu\rho} - \gamma^\nu\gamma^\tau\gamma^\rho\gamma^\mu), \\
&= 2g^{\mu\nu}\,\mathrm{Tr}(\gamma^\tau\gamma^\rho) - 2g^{\mu\tau}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho) + 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\tau) - \mathrm{Tr}(\gamma^\nu\gamma^\tau\gamma^\rho\gamma^\mu), \\
&= 8g^{\mu\nu}g^{\tau\rho} - 8g^{\mu\tau}g^{\nu\rho} + 8g^{\mu\rho}g^{\nu\tau} - \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\rho), \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\rho) &= 4\,(g^{\mu\nu}g^{\tau\rho} + g^{\mu\rho}g^{\nu\tau} - g^{\mu\tau}g^{\nu\rho})\,.
\end{aligned} \tag{A13}$$