Clifford Algebras and Spin

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ABSTRACT: This is a study notebook on Clifford algebras, rotations, spin groups, and related topics. It was written as a way of keeping up with my studies on these subjects. The discussion begins with some properties of ordinary three-dimensional rotations and proceeds towards issues, such as gimbal lock, solves them by representing rotations in terms of quaternions, and uses this as motivation to define more general Clifford algebras, spin groups, and so on. Some familiarity with topology and Lie groups is assumed.

KEYWORDS: Clifford Algebras, Rotations, Spin.

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1 Rotations in Three Dimensions

1.1 Euler Angles and Gimbal Lock

Three-dimensional rotations can be described in terms of the Lie group SO(3) (see, *exempli gratia*, Zee 2016), composed of real 3×3 matrices with unit determinant. Given a rotation matrix $R \in \mathrm{SO}(3)$, a particularly convenient way of parameterizing it for a wide range of physical applications is by means of the so-called Euler angles, as described in Goldstein 1980. In the so-called Tait–Bryan convention, common in descibring aircraft orientation, the parameterization reads (Goldstein 1980, Eq. (B-11xyz))

$$R(\phi, \theta, \psi) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\psi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\theta\cos\psi \end{pmatrix}. \tag{1.1}$$

In this choice of parameters, vanishing parameters indicate that the aircraft is horizontal. ϕ is known as the yaw, and measures whether the aircraft is "looking left or right". The pitch, θ, indicates whether it is pointing up or down. Finally, ψ, known as the roll, measures rotation around the figure axis of the aircraft. The pitch is taken to range from zero up to π, while the yaw and roll range from zero up to 2π.

Something interesting happens for $\theta = \frac{\pi}{2}$, which corresponds to an airplane in a vertical position. The matrix expression becomes, as one can check with laborious computation (or the help of `Mathematica`),

$$R\left(\phi, \frac{\pi}{2}, \psi\right) = \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\phi - \psi) & \cos(\phi - \psi) & 0 \\ \cos(\phi - \psi) & \sin(\phi - \psi) & 0 \end{pmatrix}. \tag{1.2}$$

Hence, for a vertical plane, changing the angles ϕ or ψ leads to the same result. Instead of having two degrees of freedom, as one would expect, we only have one.

Of course, this is not a problem of Physics, since there are still two possible degrees of freedom in a real situation. Given an airplane in a vertical position, we can still move it around with two degrees of freedom. A way of seeing it is by simply picking the new position of the plane and defining new Euler angles in the Tait–Bryan convention, but taking the new position of the airplane to be the point in which all angles are zero. Indicating the new angles with a prime, we'd be in a situation with $\theta' = 0$, which means $R(\phi', \theta', \psi')$ reduces to

$$R(\phi', 0, \psi') = \begin{pmatrix} \cos\phi' & \sin\phi' & 0 \\ -\cos\psi' \sin\phi' & \cos\psi' \cos\phi' & \sin\psi' \\ \sin\psi' \sin\phi' & -\sin\psi' \cos\phi' & \cos\psi' \end{pmatrix}, \tag{1.3}$$

which has two degrees of freedom. Why is there this difference?

The issue is known as gimbal lock, and was quite problematic in the Apollo 10 mission (Hanson 2006, Sec. 2.3). It crumbles down to a topological problem: the map $R(\phi, \theta, \psi)$ fails to be a local homeomorphism for some values of the angles. While it appropriately covers "small patches" of SO(3), it fails to be a global coordinate system.

Euler angles are defined on the three-torus $\mathbb{T}^3$ (or, more precisely, on the quotient space $\mathbb{T}^3/\mathbb{Z}_2$). One can show, and we will on the next section, that SO(3) is topologically equivalent to the real projective space $\mathbb{RP}^3 \simeq \mathbb{S}^3/\mathbb{Z}_2$, which is the three-sphere with antipodal points identified (analogously, the set of lines in $\mathbb{R}^4$ that cross the origin). However, the only covering space of $\mathbb{RP}^3$ is[*] $\mathbb{S}^3$. Hence, $\mathbb{T}^3/\mathbb{Z}_2$ is not a covering space of $\mathbb{RP}^3$, and as a consequence somewhere the coordinates provided by the Euler angles will fail to be a local homeomorphism. When they do, we call it gimbal lock.

1.2 Quaternions

An alternative description of rotations can be provided by employing quaternions instead of Euler angles. Our treatment follows the one given on Woit 2017, Sec. 6.2. Quaternions are a four-dimensional generalization of complex numbers which consist in equipping $\mathbb{R}^4$ with a particular product, hence turning it into an algebra. To describe the product, let us first introduce the notation

$$1 = (1,0,0,0), \quad i = (0,1,0,0), \quad j = (0,0,1,0), \quad k = (0,0,0,1). \tag{1.4}$$

In terms of this notation, the quaternionic product is defined according to (the first term is taken to be given by the lines, the second by the columns)

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \tag{1.5}$$

[*]This follows from the classification of covering spaces by means of the fundamental group. See Hatcher 2002, Theorem 1.38 or Munkres 2000, Sec. 82.

Hence, we have $ij = k$, $ki = j$, $jk = i$, and so on. Notice that multiplication is not commutative: $ij = k = -ji$. It also isn't anticommutative, for $1 \cdot i = i \cdot 1 = i$. We denote the algebra formed by $\mathbb{R}^4$ equipped with this product by $\mathbb{H}$, and call this algebra the quaternions. One can show by painful verification that this is indeed an associative algebra.

We can then represent a generic quaternion number in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{1.6}$$

for real entries q_0, q_1, q_2, and q_3.

In analogy with complex numbers, we can define a conjugation operation by

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k. \tag{1.7}$$

One can then show that, given $q, r \in \mathbb{H}$,

$$\overline{qr} = \bar{r}\bar{q}. \tag{1.8}$$

Furthermore, the Euclidean norm can be realized in terms of the conjugation operation by defining

$$|q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2. \tag{1.9}$$

Notice then that the inverse of a quaternion can be written as

$$q^{-1} = \frac{\bar{q}}{|q|^2}. \tag{1.10}$$

Let us focus on the set of quaternions with unit norm, which we'll denote by*

$$\mathrm{Sp}(1) = \{q \in \mathbb{H}; |q| = 1\}. \tag{1.11}$$

Since quaternionic multiplication is associative, so is the product of unit quaternions. 1 is a multiplicative identity for the quaternions and we've already found the inverse of all non-vanishing quaternions. Noticing at last that $|qr| = |q||r|$ we're able to conclude that $\mathrm{Sp}(1)$ is a group under quaternionic multiplication.

Notice that $\mathrm{Sp}(1)$ is build out of the quaternions with the property that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1, \tag{1.12}$$

which is exactly the equation that defines the three-sphere $\mathbb{S}^3$. Hence, topologically, $\mathrm{Sp}(1)$ corresponds to $\mathbb{S}^3$.

Our interest in the quaternions will come by means of the left group action of $\mathrm{Sp}(1)$ on $\mathbb{H}$ defined by

$$q \to uqu^{-1}, \tag{1.13}$$

*The suggestive notation hints at the fact that $\mathrm{Sp}(1)$ is related to the symplectic groups. See Hall 2015, Sec. 1.2.8.

where $q \in \mathbb{H}$, $u \in \mathrm{Sp}(1)$. For $q = q_0 \in \mathbb{R}$, this action is trivial and reduces to $q \to q$. However, it is interesting to consider it on the space of "pure imaginary" quaternions, *id est*, the quaternions of the form

$$q = q_1 i + q_2 j + q_3 k. \tag{1.14}$$

Notice these quaternions are isomorphic to $\mathbb{R}^3$ as a vector space (they do not form an algebra, for they are not closed under the quaternion product). The group action is linear on q, it does not "create" a real component (*id est*, uqu^{-1} is pure imaginary), and it has the property that

$$|q| = \left|uqu^{-1}\right|. \tag{1.15}$$

Hence, the action is a rotation*, which means we found a map Φ between $\mathrm{Sp}(1)$ and $\mathrm{SO}(3)$.

The fact that Eq. (1.13) on the preceding page is a group action implies Φ is a homomorphism. Notice it is not injective: u and $-u$ both describe the same rotation. However, it can be shown that Φ is surjective†. Φ is then what is called in the algebraic topology literature a "covering map", and we see that $\mathrm{Sp}(1)$ is a double cover of $\mathrm{SO}(3)$. Since $\mathrm{Sp}(1)$ is topologically $\mathbb{S}^3$, we see then that $\mathrm{SO}(3)$ is topologically $\mathbb{S}^3/\mathbb{Z}_2 \simeq \mathbb{RP}^3$, the three-dimensional real projective space‡.

Since the unit quaternions are topologically $\mathbb{S}^3$, their map onto the rotations does provide a covering map. Hence, they are immune to the gimbal lock problem we faced when dealing with Euler angles.

1.3 Spin(3)

A different interesting fact about the unitary quaternions $\mathrm{Sp}(1)$ occurs when one considers quantum theory. In quantum mechanics, states are defined only up to a sign. As a consequence, one is not interested exactly in representations of symmetry groups, but rather on projective representations, also known as representations up to a sign. It happens that the projective representations of $\mathrm{SO}(3)$ correspond to the true representations of its universal cover, $\mathrm{Sp}(1)$. See, *exempli gratia*, Hall 2013, Theorem 16.47 for the finite dimensional case and Hall 2013, Sec. 16.9 and Wald 1984, Sec. 13.1 for more general discussions. In these contexts, one typically favors the notation $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$. The isomorphism is shown, for example, in Woit 2017, Sec. 6.2.4.

One of the interests in writing $\mathrm{Sp}(1)$ as $\mathrm{SU}(2)$ is that it illuminates the existence of a $\mathbb{C}^2$ representation, which is a projective representation of $\mathrm{SO}(3)$. This is known as the

*The group $\mathrm{SO}(3)$ is composed of the real matrices R such that $R^\intercal R = \mathbb{1}$. It can be shown that this condition is equivalent to demanding that R is linear and $\|R\mathbf{x}\| = \|\mathbf{x}\|$ for all vectors $\mathbf{x} \in \mathbb{R}^3$.

†See Kuipers 1999, Theorem 5.1. That result establishes that the action of Eq. (1.13) on the previous page is a rotation of a certain angle around a certain axis, both of which are arbitrary. This is actually the most general form a rotation can take, a fact often named Euler's rotation theorem.

‡A more detailed discussion about covering spaces and covering maps can be found in Hatcher 2002; Munkres 2000. A more careful study of the topological properties of $\mathrm{SO}(3)$ can be found in Hall 2015, Sec. 1.3.4.

spin-$\frac{1}{2}$ representation. For more on the representation theory of SU(2) (and hence of SO(3)), see Zee 2016, Chap. IV.5.

Due to the importance of this group when studying rotations, it also receives a third name that we will later generalize for more dimensions. It is called $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1) \simeq \mathrm{SU}(2)$.

2 Clifford Algebras

Our next natural step will be to generalize the quaternions to a kind of structure known as a Clifford algebra. Our treatment is inspired by Vaz Jr. and Rocha Jr. 2016; Woit 2017.

2.1 Basic Definitions

Our definitions are based on those used in Vaz Jr. and Rocha Jr. 2016.

Definition 1 [Quadratic Space]**:**

Let V be a vector space over some field $\mathbb{F}$ (which we'll assume to be the real or complex numbers) and let $g\colon V \times V \to \mathbb{F}$ be a symmetric bilinear functional. Then the pair (V, g) is said to be a quadratic space. ♠

Definition 2 [Clifford Algebra]**:**

Let (V, g) be a quadratic space over a field $\mathbb{F}$ (which we'll assume to be the real or complex numbers). $\mathcal{A}$ be an associative algebra over $\mathbb{F}$ with unit $\mathbb{1}$ and let $\gamma\colon V \to \mathcal{A}$ be a linear transformation. $(\mathcal{A}, \gamma)$ is said to be a Clifford algebra for (V, g) if, and only if, $\mathcal{A}$ is generated as an algebra by the space $\{\gamma(\mathbf{v}); \mathbf{v} \in V\}$ and

$$\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2g(\mathbf{v}, \mathbf{u})\mathbb{1}, \tag{2.1}$$

for all $\mathbf{v}, \mathbf{u} \in V$. ♠

While we're allowing $\mathbb{F}$ to be the complex numbers, our interest will be mainly on the case $\mathbb{F} = \mathbb{R}$. In the Physics literature, Eq. (2.1) is often written in terms of the anti-commutator $\{A, B\} = AB + BA$,

$$\{\gamma(\mathbf{v}), \gamma(\mathbf{u})\} = 2g(\mathbf{v}, \mathbf{u})\mathbb{1}, \tag{2.2}$$

a notation we'll employ sometimes.

Notice that, if we pick an orthonormal basis $\{\mathbf{e}_i\}_i$of the vector space V, we get

$$\gamma(\mathbf{e}_i)\gamma(\mathbf{e}_j) + \gamma(\mathbf{e}_j)\gamma(\mathbf{e}_i) = 0, \tag{2.3}$$

and

$$\gamma(\mathbf{e}_i)^2 = g(\mathbf{e}_i, \mathbf{e}_i)\mathbb{1}. \tag{2.4}$$

Since we're working with fields of characteristic other than 2, Eq. (2.4) is actually equivalent to Eq. (2.1). Notice also that the algebra is given by

$$\mathcal{A} = \mathrm{span}\{\gamma(\mathbf{e}_1)^{\mu_1} \cdots \gamma(\mathbf{e}_n)^{\mu_n}; \mu_i = 0, 1\}, \tag{2.5}$$

where $n = \dim V$. As a consequence, notice that $\dim \mathcal{A} \leq 2^n$.

This construction is a generalization of the quaternions and other known structures. Let us see some examples (adapted from Woit 2017, Sec. 28.2).

Example [Complex Numbers]**:**

Pick $V = \mathbb{R}$ and g given by $g(\mathbf{e}_1, \mathbf{e}_1) = -1$. Then we can pick a Clifford algebra given by $\mathcal{A} = \mathbb{C}$ with $\gamma(\mathbf{e}_1) = i$. Hence, the complex numbers correspond to a Clifford algebra. ♥

Example [Quaternions]**:**

Pick $V = \mathbb{R}^2$ and g given by $g(\mathbf{e}_i, \mathbf{e}_j) = -\delta_{ij}$, where δ_{ij} is the Kronecker delta. Then we can pick $\mathcal{A} = \mathbb{H}$ with $\gamma(\mathbf{e}_1) = i$ and $\gamma(\mathbf{e}_2) = j$. One will have $\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2) = k$, and it can be checked that the construction does yield a Clifford algebra. ♥

Example [2×2 Real Matrices]**:**

Pick $V = \mathbb{R}^2$ and g given by $g(\mathbf{e}_i, \mathbf{e}_j) = -(-1)^i \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then we can pick $\mathcal{A} = \mathbb{M}_2$, the space of 2×2 real matrices, with

$$\gamma(\mathbf{e}_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma(\mathbf{e}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

It can be checked that the construction does yield a Clifford algebra. ♥

Our focus will be on the so-called universal Clifford algebras, which correspond to the Clifford algebras with $\dim \mathcal{A} = 2^n$. More details can be found at Vaz Jr. and Rocha Jr. 2016, Chap. 3. It can be shown that the universal Clifford algebra for (V, g) is unique up to isomorphism, so we'll denote it by $\mathcal{C}\ell(V, g)$.

More specifically, we'll be interested in the cases with $V = \mathbb{R}^n$. If in its diagonal form g has r positive signs and $s = n - r$ negative signs, we'll simply denote its universal Clifford algebra by $\mathcal{C}\ell_{r,s}$. For the particular case $r = n$ ($s = 0$), we'll simply write $\mathcal{C}\ell_n$.

Notice that our previous examples show that $\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$, $\mathcal{C}\ell_{0,2} \simeq \mathbb{H}$, $\mathcal{C}\ell_{1,1} \simeq \mathbb{M}_2$.

2.2 Generalizing the Quaternions

To understand how Clifford algebras are related to rotations, we'll follow the construction done in Woit 2017, Sec. 29.2.1. For starters, let us begin by stating the so-called Cartan–Dieudonné theorem.

Theorem 3 [Cartan–Dieudonné]**:**

Let (V, g) be a quadratic space and $\mathrm{O}(V, g)$ the groups of its isometries, id est, *its "rotation group". Then given $R \in \mathrm{O}(V, g)$, R can be written as the product of at most* $\dim V$ *simple reflections.* □

Proof:

See Garling 2011, Sec. 4.8. ■

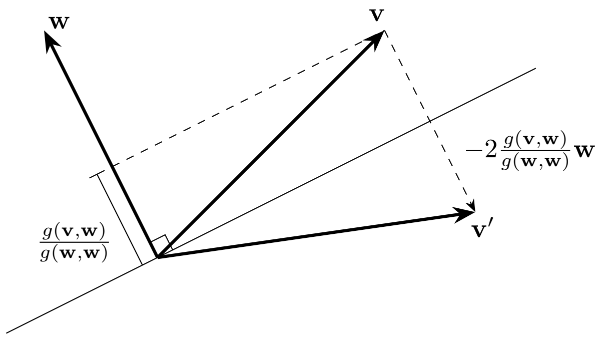

Figure 1: *Depiction of how a reflection with respect to the plane perpendicular to* $\mathbf{w}$ *corresponds to Eq. (2.7). Adapted from Woit 2017, Figure 29.1, p. 377.*

Our interest in this result is due to the fact that it is particularly simple to write reflections in terms of a Clifford algebra. As an illustration, we'll work with the algebras $\mathcal{C}\ell_n$.

Pick some vector $\mathbf{w} \in \mathbb{R}^n$. Orthogonal reflection with respect to the plane perpendicular to $\mathbf{w}$ corresponds to the mapping

$$\mathbf{v} \to \mathbf{v}' = \mathbf{v} - 2\frac{g(\mathbf{v},\mathbf{w})}{g(\mathbf{w},\mathbf{w})}\mathbf{w}. \tag{2.7}$$

This is pictured on Fig. 1.

We may then apply the linear transformation γ that is part of the Clifford algebra to Eq. (2.7). This leads to

$$\gamma(\mathbf{v}) \to \gamma(\mathbf{v}') = \gamma(\mathbf{v}) - 2\frac{g(\mathbf{v},\mathbf{w})}{g(\mathbf{w},\mathbf{w})}\gamma(\mathbf{w}), \tag{2.8a}$$

$$= \gamma(\mathbf{v}) - (\gamma(\mathbf{v})\gamma(\mathbf{w}) + \gamma(\mathbf{w})\gamma(\mathbf{v}))\frac{\gamma(\mathbf{w})}{g(\mathbf{w},\mathbf{w})}. \tag{2.8b}$$

Notice that, if $\mathbf{w} \neq \mathbf{0}$, then

$$\gamma(\mathbf{w})\frac{\gamma(\mathbf{w})}{g(\mathbf{w},\mathbf{w})} = \mathbb{1}. \tag{2.9}$$

Therefore,

$$\gamma(\mathbf{w})^{-1} = \frac{\gamma(\mathbf{w})}{g(\mathbf{w},\mathbf{w})}, \tag{2.10}$$

which is analogous to the expression we had for quaternions. Using this expression on Eq. (2.8) we are able to see that

$$\gamma(\mathbf{v}) \to \gamma(\mathbf{v}') = \gamma(\mathbf{v}) - (\gamma(\mathbf{v})\gamma(\mathbf{w}) + \gamma(\mathbf{w})\gamma(\mathbf{v}))\gamma(\mathbf{w})^{-1}, \tag{2.11a}$$

$$= \gamma(\mathbf{v}) - \gamma(\mathbf{v}) - \gamma(\mathbf{w})\gamma(\mathbf{v})\gamma(\mathbf{w})^{-1}, \tag{2.11b}$$

$$= -\gamma(\mathbf{w})\gamma(\mathbf{v})\gamma(\mathbf{w})^{-1}. \tag{2.11c}$$

Hence, a single reflection can be written as a conjugation. Since this is a reflection, *id est*, it has negative determinant, it is an element of $\mathrm{O}(n)$, but not of $\mathrm{SO}(n)$. To get an element of $\mathrm{SO}(n)$ we'll need to perform a second reflection to get an expression of the form

$$\gamma(\mathbf{v}) \to \gamma(\mathbf{v}') = -\gamma(\mathbf{w}_2)\big(-\gamma(\mathbf{w}_1)\gamma(\mathbf{v})\gamma(\mathbf{w}_1)^{-1}\big)\gamma(\mathbf{w}_2)^{-1}, \tag{2.12a}$$

$$= (\gamma(\mathbf{w}_2)\gamma(\mathbf{w}_1))\gamma(\mathbf{v})(\gamma(\mathbf{w}_2)\gamma(\mathbf{w}_1))^{-1}. \tag{2.12b}$$

In analogy with our previous definition of $\mathrm{Spin}(3)$ in terms of quaternions, we now define $\mathrm{Spin}(n)$.

Definition 4 [Pin and Spin]**:**
We define the group $\mathrm{Spin}(n)$ to be the group composed of invertible elements of $\mathcal{C}\ell_n$ of the form $\gamma(\mathbf{w}_1)\cdots\gamma(\mathbf{w}_{2k})$, $2k \leq n$, where the vectors $\mathbf{w}_i$ satisfy $g(\mathbf{w}_i, \mathbf{w}_i) = 1$. The group product is given by the Clifford algebra product.

A bit more generally, we may define the group $\mathrm{Pin}(r, s)$ to be the group composed of of $\mathcal{C}\ell_{r,s}$ of the form $\gamma(\mathbf{w}_1)\cdots\gamma(\mathbf{w}_k)$, $k \leq r+s$, where the vectors $\mathbf{w}_i$ satisfy $g(\mathbf{w}_i, \mathbf{w}_i) = \pm 1$. $\mathrm{Spin}(r, s)$ is then defined to be the even part of $\mathrm{Pin}(r, s)$, *id est*, it has the same definition, but only for even k*. ♠

It is, of course, interesting to now consider how we can recover $\mathrm{Spin}(3)$ from this new formalism.

Example [Recovering Three-Dimensional Rotations]**:**
$\mathcal{C}\ell_3$ is actually isomorphic to $\mathbb{M}_2(\mathbb{C})$ as a *real* algebra. To see this, notice that $\mathcal{C}\ell_3$ is characterized by $\{\gamma(\mathbf{e}_i), \gamma(\mathbf{e}_j)\} = 2\delta_{ij}\mathbb{1}$, where δ_{ij} is the three-dimensional Kronecker delta. Coincidentally, the Pauli matrices satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$. We may then pick $\gamma(\mathbf{e}_i) = \sigma_i$. The products of Pauli matrices do span the whole $\mathbb{M}_2(\mathbb{C})$ algebra.

Let us then proceed to study the $\mathrm{Spin}(3)$ group. We begin by noticing that the even elements of $\mathcal{C}\ell_3$, *id est*, the elements that are written as a product of an even number of vectors, are given by

$$\mathcal{C}\ell_3^+ = \mathrm{span}\{\mathbb{1}, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\}, \tag{2.13}$$

where $\mathcal{C}\ell_3^+$ stands for the even part of $\mathcal{C}\ell_3$. Recalling that $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$, where ϵ_{ijk} is the Levi-Civita symbol, we can also write this expression as

$$\mathcal{C}\ell_3^+ = \mathrm{span}\{\mathbb{1}, i\sigma_3, i\sigma_1, i\sigma_2\}. \tag{2.14}$$

However, this is isomorphic to the quaternions. Indeed, we can obtain an isomorphism by mapping

$$1 \to \mathbb{1}, \quad i \to -i\sigma_1, \quad j \to -i\sigma_2, \quad k \to -i\sigma_3. \tag{2.15}$$

Hence, what we've learned so far is that there is an isomorphim between elements of $\mathcal{C}\ell_3$ of the form $\gamma(\mathbf{w}_1)\cdots\gamma(\mathbf{w}_{2k})$, $2k \leq 3$, and the quaternions. We now have to show that

*This definition of $\mathrm{Pin}(N)$ and $\mathrm{Spin}(N)$ is inspired by Woit 2017's definition of $\mathrm{Spin}(n)$ and based on Lawson and Michelsohn 1989's definition of $\mathrm{Pin}(N)$ and $\mathrm{Spin}(N)$. Nevertheless, it is good to check whether our definition is indeed equivalent to that of Lawson and Michelsohn 1989.

the imposition that $g(\mathbf{w}_i, \mathbf{w}_i) = 1$ restricts the isomorphism to the unit quaternions. To see this, let us denote the i, j, and k elements of the quaternions by ε_1, ε_2, and ε_3. In this case, we notice that the most general element of the form $\gamma(\mathbf{w}_1)\cdots\gamma(\mathbf{w}_{2k})$, $2k \leq 3$, with $g(\mathbf{w}_i, \mathbf{w}_i) = 1$ can be written in components as

$$v_i w_j \sigma_i \sigma_j = v_i w_j \delta_{ij} \mathbb{1} + i v_i w_j \epsilon_{ijk} \sigma_k. \tag{2.16}$$

Using our isomorphism with the quaternions, we see this expression becomes

$$v_i w_j \sigma_i \sigma_j \to v_i w_j \delta_{ij} - v_i w_j \epsilon_{ijk} \varepsilon_k. \tag{2.17}$$

Computing the quaternion norm of this expression leads to

$$|v_i w_j \delta_{ij} - v_i w_j \epsilon_{ijk} \varepsilon_k|^2 = (\mathbf{v} \cdot \mathbf{w})^2 + \|\mathbf{v} \times \mathbf{w}\|^2, \tag{2.18a}$$

$$= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \left(\cos^2\theta + \sin^2\theta\right), \tag{2.18b}$$

$$= 1, \tag{2.18c}$$

where θ is the angle between the unit vectors $\mathbf{v}$ and $\mathbf{w}$. Hence, the norm condition we imposed does lead to the condition that the quaternion norm is 1.

Therefore, our generalized definition of $\mathrm{Spin}(n)$ does obtain $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1)$, as expected.

An alternative proof would be to notice that Eq. (2.16) defines an element of $\mathrm{SU}(2)$, and hence $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1) \simeq \mathrm{SU}(2)$. ♥

2.3 Some Results

Knowing the definitions of Clifford algebras and the spin groups, it is interesting for us to point out a few results on them. We refer to more specialized texts for the more important theorems and restrict ourselves to a few ones that were interesting for the author when writing this notebook. We also refer to the specialized literature for the proofs.

Let us begin by defining the even part of a Clifford algebra. We'll do it by following the particular case we used when discussing three-dimensional rotations.

Definition 5 [Even part of a Clifford algebra]**:**

Let (V, g) be a quadratic space. We define the even part of $\mathcal{C}\ell(V, g)$, denoted $\mathcal{C}\ell^+(V, g)$, as the subalgebra of $\mathcal{C}\ell(V, g)$ generated by elements that are products of an even number of generators, *id est*, if $\{\mathbf{e}_i\}_i$ is an orthonormal basis of V, we have

$$\mathcal{C}\ell^+(V, g) = \mathrm{span}\left\{\gamma(\mathbf{e}_1)^{\mu_1} \cdots \gamma(\mathbf{e}_n)^{\mu_n}; \mu_i = 0, 1, \sum_i \mu_i \text{ is even}\right\}, \tag{2.19}$$

where $n = \dim V$ (*conferatur* Eq. (2.5) on page 5). For the Clifford algebras $\mathcal{C}\ell_{r,s}$, we'll simply write the even part as $\mathcal{C}\ell^+_{r,s}$. ♠

It is worth mentioning that the definitions given, *exempli gratia*, in Vaz Jr. and Rocha Jr. 2016; Lawson and Michelsohn 1989; Garling 2011 are a bit more elaborate, but consist of the same idea when dealing with problems in finite dimension*.

*At least as far as I know.

Theorem 6:
The isomorphisms

$$\mathcal{C}\ell_{r,s}^{+} \simeq \mathcal{C}\ell_{s,r-1} \simeq \mathcal{C}\ell_{r,s-1} \simeq \mathcal{C}\ell_{s,r}^{+} \tag{2.20}$$

hold. □

Proof:
See Vaz Jr. and Rocha Jr. 2016, Theorem 4.4. ■

Due to the very definition of $\mathrm{Spin}(n)$, it holds that $\mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n^+$. What Theorem 6 tells us is that $\mathrm{Spin}(n) \subseteq \mathcal{C}\ell_{0,n-1}$, which lets us that we can obtain information about $\mathrm{Spin}(n)$ by looking simply at $\mathcal{C}\ell_{0,n-1}$.

An specific example if that of $n = 3$. We already know that $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1)$, the unit quaternions. Furthermore, we've seen earlier that $\mathcal{C}\ell_{0,2} \simeq \mathbb{H}$. Hence, we have $\mathrm{Spin}(3) \subseteq \mathcal{C}\ell_{0,2}$, as expected.

It is also interesting to consider the analogous result for complex Clifford algebras. If the vector space we're picking is $V = \mathbb{C}^n$, then any symmetric bilinear form can be written as δ_{ij} in an appropriate basis, so the Clifford algebras $\mathcal{C}\ell(\mathbb{C}^n, g)$ can be denoted simply as $\mathbb{C}\ell_n$. It can be shown (Vaz Jr. and Rocha Jr. 2016, Theorem 4.2) that $\mathbb{C}\ell_n \simeq \mathbb{C} \otimes \mathcal{C}\ell_{r,s}$, where $r + s = n$, regardless of the particular values of r and s. Theorem 6 then yields

Theorem 7:
The isomorphism

$$\mathbb{C}\ell_n^{+} \simeq \mathbb{C}\ell_{n-1} \tag{2.21}$$

hold. □

This implies, in particular, that to understand $\mathrm{Spin}(n)$ we may study $\mathbb{C}\ell_{n-1}$. This is particularly interesting for applications in quantum theory, when one typically has to deal with complex variables.

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