As Time Goes By: Seeking a Solution to Maxwell's Equations

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(Dated: January 31, 2019)

The power of potential formulation of Electro and Magnetostatics is widely known by Physics students. It allows us to calculate the scalar potential in order to find the Electric Field and then use the mathematical similarities between Electro and Magnetostatics to find the vector potential and, then, the Magnetic Field. In this paper, aimed at undergraduate students, we explore the potential formulation of Electrodynamics, departing from static scalar and vector potentials and walking towards the potential expressions for electrodynamical fields. Using the Lorenz gauge, we find the retarded potentials and Jefimenko's Equations, the latter being the solution to the microscopic version of Maxwell's Equations.

Keywords: time-dependent electromagnetic fields, retarded potentials, Jefimenko's Equations

I. OUTLINE

Undergraduates are, in general, familiar with the potential formulation of Electro and Magnetostatics and are used to the proceedings related to finding the physical fields **E** and **B** by differentiating the auxiliary fields V and **A**, which are, in general, far easier to obtain through integration. This formulation, including the Helmholtz Theorem, is reviewed at Section II, while the time-dependent formulation is covered at Section III.

Having the differential equations for V and **A** in terms of the charge and current densities ρ and **J**, gauge freedom is discussed at Section IV, in order to express the differential equations in the Lorenz gauge, allowing an intuitive introduction of retarded potentials at Section V. In this section, it is also proved that the retarded potentials are not only intuitive, but the true solutions to the wave equations found at Section IV.

Section VI proposes a solution to Maxwell's Equations through a reasoning similar to the one which has led us before to the retarded potentials, highling the difficulties inherent to the problem.

After some problems have been encountered at the previous section, Section VII retraces the steps taken in order to find the retarded potentials and makes the appropriate adaptations to find the correct expressions for the physical fields - namely, Jefimenko's Equations.

Finally, the Appendix contains some proof sketches for two vector calculus theorems which were necessary at Section VII.

It is assumed that the reader is familiar with vector calculus, Maxwell's Equations and the Continuity Equation.

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II. INTRODUCTION

The main goal of Electrostatics and Magnetostatics is to obtain the electromagnetic fields **E** and **B** (which, for now, we are supposing to be time-independent) in order to calculate the movement of electric charges under their action through the mechanisms of Classical Mechanics and Lorentz Force, which is given by

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{1}$$

However, such a direct approach isn't always something easy to calculate, and it gets even more cumbersome when we allow the fields to change as time passes. Therefore, a more common approach is to use Helmholtz Theorem in order to change the problem slightly. This Theorem is stated[1] and proved[2] as following:

Helmholtz Theorem. *If the divergence* $(\boldsymbol{\nabla}\cdot\mathbf{F})(\mathbf{r})$ *and the curl* $(\boldsymbol{\nabla}\times\mathbf{F})(\mathbf{r})$ *of a vector function* $\mathbf{F}(\mathbf{r})$ *are specified, and if they both go to zero faster than* $1/r^2$ *as* $r \to +\infty$*, and if* $\mathbf{F}(\mathbf{r})$ *goes to zero as* $r \to +\infty$*, then* $\mathbf{F}$ *is given uniquely by*

$$\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W} \tag{2}$$

where U *and* $\mathbf{W}$ *are given by*

$$U(\mathbf{r}) \equiv \frac{1}{4\pi}\int \frac{(\boldsymbol{\nabla}\cdot\mathbf{F})(\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|}\,\mathrm{d}\tau' \tag{3}$$

and

$$\mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi}\int \frac{(\boldsymbol{\nabla}\times\mathbf{F})(\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|}\,\mathrm{d}\tau'. \tag{4}$$

□

Proof. It is known from vector calculus that

$$-\nabla^2\mathbf{Z} = -\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{Z}) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{Z}). \tag{5}$$

Suppose that we may write $\mathbf{F}$ as

$$\mathbf{F} = -\nabla^2 \mathbf{Z}, \tag{6}$$

for some vector function $\mathbf{Z}$. Equation (6) is usually known as Poisson's Equation. If $\mathbf{F}$ vanishes fast enough[1] as $r \to +\infty$, there is indeed such $\mathbf{Z}$ and it is given by[2][1, 2]

$$\mathbf{Z} = \frac{1}{4\pi} \int \frac{\mathbf{F}(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, . \tag{7}$$

Indeed, if we apply the Laplacian on both sides of equation (7), we see that, since the differentiation is done with respect to the unprimed coordinates and the integral with respect to the primed coordinates,

$$\begin{aligned} \nabla^2 \mathbf{Z} &= \frac{1}{4\pi} \int \nabla^2 \left(\frac{\mathbf{F}(\mathbf{r}')}{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= \frac{1}{4\pi} \int \mathbf{F}(\mathbf{r}') \nabla^2 \left(\frac{1}{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= -\frac{1}{4\pi} \int \mathbf{F}(\mathbf{r}') \cdot 4\pi \delta^3 \left(\boldsymbol{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= -\mathbf{F}(\mathbf{r}). \end{aligned} \tag{8}$$

In equation (8), δ^3 is the three-dimensional Dirac delta "function"[3].

Now, equation (5) reads

$$\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W}, \tag{9}$$

for U and $\mathbf{W}$ given by

$$U = \boldsymbol{\nabla} \cdot \mathbf{Z}, \quad \mathbf{W} = \boldsymbol{\nabla} \times \mathbf{Z}. \tag{10}$$

Since all curls are divergenceless, we have that

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} &= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} U), \\ &= -\nabla^2 U, \end{aligned} \tag{11}$$

which is once more Poisson's Equation[4]. Thus, we have that

$$U(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\left(\boldsymbol{\nabla} \cdot \mathbf{F}\right)(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, , \tag{12}$$

which would be the solution for one of the coordinates of equation (6).

Gradients being irrotational, a very similar reasoning allows us to see that, due to equation (5),

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{W}), \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{W}) - \nabla^2 \mathbf{W}, \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{Z})) - \nabla^2 \mathbf{W}, \\ &= -\nabla^2 \mathbf{W}, \end{aligned} \tag{13}$$

which is again Poisson's Equation.

Finally,

$$\mathbf{W}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\left(\boldsymbol{\nabla} \times \mathbf{F}\right)(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, . \tag{14}$$

This concludes the proof. ■

In the above theorem, and through the rest of this paper, $\mathrm{d}\tau'$ represents the volume element associated with the primed coordinates, which are themselves the coordinates of source charges and currents (see Fig. 1). The conventions used herein are the same ones used in [1].

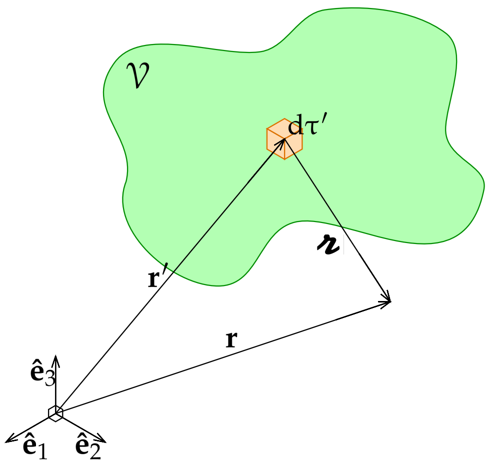

Figure 1. The primed coordinates are the ones in which the integrals are meant to be taken, *i.e.*, the charges' and currents' (represented by the blob) coordinates. The unprimed coordinates are where we evaluate the field's values. The vector $\boldsymbol{\mathscr{r}}$ represents the distance between both.

When dealing with static electromagnetic problems, it might be easier to deal directly with the special case when either the divergence or the curl is zero. Then, we may write Helmholtz Theorem in the following way:

Theorem 1. *Let* $\mathbf{E}$ *be a curl-less field and let* $\mathbf{B}$ *be a divergenceless field. Then we may write them as*

$$\mathbf{E} = -\boldsymbol{\nabla} V, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{15}$$

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[1] This being a Physics paper, we are not too interested in the formal procedures involved in this development. Further works shall fill in these gaps with proper mathematical rigour.

[2] The "$\boldsymbol{\mathscr{r}}$" notation should be understood in the following way: $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$, $\mathscr{r} \equiv \|\boldsymbol{\mathscr{r}}\|$, $\hat{\boldsymbol{\mathscr{r}}} \equiv \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}}$.

[3] For more information on this "thing" (which is actually not a function) or the divergence of $\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$, take a look at [1]'s first chapter.

[4] Although equation (6) involves vectors and equation (11) involves scalars, both are called Poisson's Equation. After all, the former is just three copies of the latter, one in each coordinate.

where V and $\mathbf{A}$ are given by

$$V(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \cdot \mathbf{E})(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \tag{16}$$

and

$$\mathbf{A}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla} \times \mathbf{B})(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, . \tag{17}$$

□

Under this new statement, it will soon become clear that we can get a lot of information on the electromagnetic fields through their potentials. This is due to Maxwell's Equations on the static case:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} & \text{(Gauss's Law)}, \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 & \text{(Magnetic Gauss's Law)}, \\ \boldsymbol{\nabla} \times \mathbf{E} = 0 & \text{(Faraday's Law)}, \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} & \text{(Ampère's Law)}. \end{cases} \tag{18}$$

From equations (16), (17) and (18), we see that we can write

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, , \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, \mathrm{d}\tau' \, . \tag{19}$$

These are the so called (static) scalar and vector potentials (as the names might suggest, the scalar potential is V and the vector potential is $\mathbf{A}$). Having calculated these potentials, one might obtain $\mathbf{E}$ and $\mathbf{B}$ through equation (15). With the aid of vector calculus, it is seen that[5]

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} V, \\ &= -\frac{1}{4\pi\epsilon_0} \int \boldsymbol{\nabla} \left(\frac{\rho(\mathbf{r}')}{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= -\frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \cdot \boldsymbol{\nabla} \left(\frac{1}{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, \mathrm{d}\tau' \, , \end{aligned}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \cdot \hat{\mathscr{r}} \, \mathrm{d}\tau' \, . \tag{20}$$

Equation (20) is widely know as Coloumb's Law, and its application to a single point charge, together with the Lorentz Force Law (equation (1)), will give rise to the familiar inverse square force law.

A similar calculation with the magnetic equation in equation (15) shows that

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}, \\ &= \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \times \left(\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \right) \mathrm{d}\tau' \, , \\ &= \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} \cdot \boldsymbol{\nabla} \times \mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}') \times \boldsymbol{\nabla} \left(\frac{1}{\mathscr{r}} \right) \right] \mathrm{d}\tau' \, , \\ &= \frac{\mu_0}{4\pi} \int \left[\mathbf{0} + \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right] \mathrm{d}\tau' \, , \end{aligned}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, \mathrm{d}\tau' \, . \tag{21}$$

Equation (21) is the magnetic equivalent to Coulomb's Law and it is known as Biot-Savart Law. When we are talking about fields that vanish when $r \to +\infty$, it is safe to say that these laws solve Maxwell's Equations for the electromagnetic fields. Therefore, our objective when seeking a time-dependent solution is to find a more general form of these very same laws.

However, before we keep going, there is a final remark which should be done within this section. By taking equation (15) and replacing it in Gauss's Law, we can see that

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} V) &= -\frac{\rho}{\epsilon_0}, \\ \nabla^2 V &= -\frac{\rho}{\epsilon_0}. \end{aligned} \tag{22}$$

Similarly, by replacing equation (15) in Ampère's Law we obtain

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \mu_0 \mathbf{J}, \\ \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \mu_0 \mathbf{J}, \\ \nabla^2 \mathbf{A} &= -\mu_0 \mathbf{J}, \end{aligned} \tag{23}$$

where we used equation (5). Besides, we chose the divergence of $\mathbf{A}$ to be zero. There is no problem in doing that, since we are only interested in its curl, which remained unchanged.

Equations (22) and (23) are, once again, Poisson's Equation. Their solutions are given by equation (19).

[5] It is important to notice that while the integrals are meant to be taken on the primed coordinates, the derivatives are done with respect to the unprimed coordinates.

III. TIME-DEPENDENT POTENTIALS

Now we are ready to explore Maxwell's Equations in their full glory, *i.e.*,

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{cases} \tag{24}$$

From hereon, when we speak of the names previously mentioned at equation (18), we will be talking about these versions of them. The only exception is the equation for the curl of $\mathbf{B}$, which we shall call Ampère-Maxwell Law.

Since Faraday's Law is no longer homogeneous, we can't, for now, use Theorem 1 to write $\mathbf{E}$ in terms of potentials. However, the Magnetic Gauss's Law has remained unchanged, hence it is still true that

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{25}$$

though the expression we previously had for $\mathbf{A}$ doesn't need to be valid anymore. We might then write

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{A}\right), \\ \nabla \times \mathbf{E} &= -\nabla \times \frac{\partial \mathbf{A}}{\partial t}, \\ \nabla \times \mathbf{E} + \nabla \times \frac{\partial \mathbf{A}}{\partial t} &= \mathbf{0}, \\ \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) &= \mathbf{0}, \\ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} &= -\nabla V, \\ \mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \end{aligned} \tag{26}$$

We have obtained equations (25) and (26) through the use of only the Magnetic Gauss's Law and of Faraday's Law, which means we still have some cards up our sleeves. Using Gauss's Law and equation (26) it follows that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= -\nabla \cdot (\nabla V) - \nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t}\right), \\ \nabla^2 V + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}\right) &= -\frac{\rho}{\epsilon_0}. \end{aligned} \tag{27}$$

Likewise, adding the Ampère-Maxwell Law to our information we see that

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\nabla V + \frac{\partial \mathbf{A}}{\partial t}\right), \\ \nabla \times (\nabla \times \mathbf{A}) &= \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}, \\ \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}, \\ \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \mu_0 \epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) &= \mu_0 \mathbf{J}, \\ \left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) &= -\mu_0 \mathbf{J}. \end{aligned} \tag{28}$$

It is very pleasing to see that equations (27) and (28) return to equations (22) and (23) when we pick V and $\mathbf{A}$ to be constant over time and $\nabla \cdot \mathbf{A} = 0$. However, the "updated" equations seem to be much harder to be solved, which is definitely not pleasing. Can we write them in a simpler way?

IV. GAUGE FREEDOM

In Classical Mechanics, we are allowed to choose the reference system in which we are going to work out some problem. This means we may apply, as an example, spatial translations, time translations, rotations and/or boosts to the system in which we are interested without any loss of information, if the system is symmetric with respect to that transformation. What if we could also change "reference systems" in Electrodynamics?

In fact, we already perfomed such a thing in this paper. In Section II, we chose the divergence of $\mathbf{A}$ to be zero, since we were only interested in its curl. Currently, we also need to be careful about the time-derivative of $\mathbf{A}$, so can we still choose its divergence?

In order to answer this question, let's take a look on the way these fields should transform. Let

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathbf{a}, \quad V \to V' = V + b. \tag{29}$$

Since $\mathbf{A}'$ and V' must describe the very same physical fields as $\mathbf{A}$ and V, it follows from equation (25) that

$$\begin{aligned} \nabla \times \mathbf{A} &= \nabla \times \mathbf{A} + \nabla \times \mathbf{a}, \\ \nabla \times \mathbf{a} &= \mathbf{0}. \end{aligned} \tag{30}$$

Furthermore, we have, this time from equation (26), that

$$\begin{aligned} -\nabla V - \frac{\partial \mathbf{A}}{\partial t} &= -\nabla V - \nabla b - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{a}}{\partial t}, \\ \nabla b &= -\frac{\partial \mathbf{a}}{\partial t}. \end{aligned} \tag{31}$$

However, since $\boldsymbol{\nabla} \times \mathbf{a} = \mathbf{0}$, we might just apply Theorem 1 to obtain $\mathbf{a} = \boldsymbol{\nabla}\chi$, for some scalar function χ. With this in our hands, equation (31) now reads

$$\begin{aligned}
\boldsymbol{\nabla} b + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\chi\right) &= \mathbf{0}, \\
\boldsymbol{\nabla}\left(b + \frac{\partial \chi}{\partial t}\right) &= \mathbf{0}, \\
\int \boldsymbol{\nabla}\left(b + \frac{\partial \chi}{\partial t}\right) \cdot \mathrm{d}\mathbf{l} &= \int \mathbf{0} \cdot \mathrm{d}\mathbf{l}\,, \\
b + \frac{\partial \chi}{\partial t} &= \kappa(t), \\
b &= -\frac{\partial \chi}{\partial t} + \kappa(t).
\end{aligned} \tag{32}$$

In the previous calculation, it was used (perhaps in an indirect way) the known fact from vector calculus that

$$\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla}\mathbf{F} \cdot \mathrm{d}\mathbf{l} = \mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a}). \tag{33}$$

Since the spatial gradient of any function $\kappa(t)$ is certainly going to be $\mathbf{0}$, we might simply consider κ as part of χ[3] and finaly write the general rule[6] for the potential transformations:

$$\begin{cases} \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi, \\ V \to V - \dfrac{\partial \chi}{\partial t}. \end{cases} \tag{34}$$

The transformation shown in equation (34) is said to be a gauge transformation[3], while χ - which may depend on $\mathbf{r}$ and t - is said to be a gauge function[3].

Applying the divergence operator to the vector potential's tranformation in equation (34) shows that

$$\boldsymbol{\nabla} \cdot \mathbf{A} \to \boldsymbol{\nabla} \cdot \mathbf{A} + \nabla^2 \chi, \tag{35}$$

which essentialy means we might choose the divergence of $\mathbf{A}$ to be the one which pleases us the most[4].

The gauge we chose earlier when dealing with static problems, in which $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, is named Coulomb's gauge. In this gauge, equations (27) and (28) read

$$\begin{cases} \nabla^2 V = -\dfrac{\rho}{\epsilon_0}, \\ \left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \mu_0\epsilon_0 \boldsymbol{\nabla}\left(\dfrac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}. \end{cases} \tag{36}$$

Though the equation for the scalar potencial has been reduced to a simple Poisson's Equation, the equation for $\mathbf{A}$ is still pretty cumbersome and does not quite interest us right now.

Equation (28) already resembles a wave equation, but it would match it perfectly if it was possible for us to get rid of the gradient's argument. Therefore, it is exactly what we are going to do: pick $\boldsymbol{\nabla} \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$. This choice is called Lorenz gauge, in which equations (27) and (28) read

$$\begin{cases} \nabla^2 V - \mu_0\epsilon_0 \dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}, \\ \nabla^2 \mathbf{A} - \mu_0\epsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \end{cases} \tag{37}$$

Equation (37) shows us that these potentials respect the three-dimensional wave equation and travel at a finite speed $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. Moreover, we also know these waves are generated by the charges ρ and currents $\mathbf{J}$. It is now possible to explore this new information in order to update Coulomb's and Biot-Savart Laws to time-dependent cases.

V. RETARDED POTENTIALS

Since we know the fields travel at a finite speed, which we called c, and are generated by the charges and currents, perhaps evaluating these quantities in the past would give us the potentials in the present.

Think about it: since the potentials travel at a speed c, it means that $V(\mathbf{r}, t)$ was actually caused by the charge distribution at a time $t_r = t - \frac{\mathscr{r}}{c}$. Therefore, it seems to be reasonable to guess that the potentials can be found by evaluating equations (19) without keeping time fixed, but instead considering the past charge distributions[7]. In other words, we could attempt to solve equation (37) by guessing that its solution should be given by

$$\begin{cases} V(\mathbf{r}, t) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right)}{\mathscr{r}}\, \mathrm{d}\tau' \\ \mathbf{A}(\mathbf{r}, t) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\mathbf{J}\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right)}{\mathscr{r}}\, \mathrm{d}\tau' \end{cases} \tag{38}$$

[6] Actually, the previous argument is not enough to state this is a general transformation rule. As the Helmholtz Theorem might suggest, only the curl of $\mathbf{a}$ is not enough to state it might be written as the gradient of a function, as an example. However, it is fairly easy to check that the transformation here shown is indeed possible.

[7] One should notice that it isn't a single instant in time that is being considered for all space, but actually a different instant for each point. The idea is something similar to taking a panoramic picture of a moving object: you get a distorted photograph of the whole thing. This is due to the fact that, since information is travelling at a finite speed, each point in space needs a different time for its information to arrive at the place in which you are evaluating the fields.

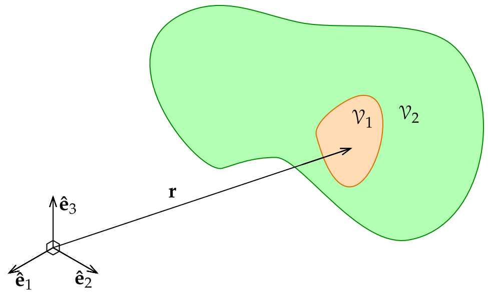

Figure 2. We split the space in which we are integrating into two disjoint sets: $\mathcal{V}_1$ and $\mathcal{V}_2$. This division is such that the point in which we evaluate the fields, $\mathbf{r}$, is always in $\mathcal{V}_1$. If our integration limits didn't include $\mathbf{r}$, there is no problem: we can always integrate over all space by setting ρ and $\mathbf{J}$ as zero on the outside of our original integration volume.

Of course, it isn't enough to simply say "this seems like a reasonable solution, so let's just stick with it", for physical intuition can trick us into false results (as we shall see later in Section VI). Therefore, we must check that equation (38) does satisfy equation (37). The following proof is due to Riemann[5] and uses a clever and indirect argument to re-obtain equation (37) from equation (38). The same result can be found through direct calculation by simply substituting equation (38) in equation (37)[1].

The idea behind Riemann's proof is to separate the source term from the homogeneous wave equation and them adding them up back again. This might be accomplished if we split the volume in which we are integrating (which we shall call $\mathcal{V}$ from hereon) into two disjoint volumes, say $\mathcal{V}_1$ and $\mathcal{V}_2$ (see Fig. 2). In mathematical notation, $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$[8]. Besides, since V is the potential obtained when integrating over $\mathcal{V}$, we define the "partial potentials" V_1 and V_2 by

$$V_i(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_i} \frac{\rho\left(\mathbf{r}',t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\,. \tag{39}$$

Notice that we have then that

$$\begin{aligned} V(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_1\sqcup\mathcal{V}_2} \frac{\rho\left(\mathbf{r}',t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\,,\\ &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_1} \frac{\rho\left(\mathbf{r}',t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\\ &\quad+ \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2} \frac{\rho\left(\mathbf{r}',t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\,,\\ &= V_1(\mathbf{r},t) + V_2(\mathbf{r},t). \end{aligned} \tag{40}$$

[8] The $\sqcup$ (called disjoint union) notation means, in this case, that $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$.

Finally, let's impose that $\mathbf{r} \in \mathcal{V}_1$, *i.e.*, the point in which we are evaluating V is always in $\mathcal{V}_1$.

Suppose now that we let $\mathcal{V}_1$ be very small, small enough for the time correction to be negligible (don't worry, we will take a limit later on). Then it is true that

$$\begin{aligned} \mathbf{r} \in \mathcal{V}_1 &\Rightarrow \rho\left(\mathbf{r},t_r\right) \to \rho\left(\mathbf{r},t\right),\\ V_1(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_1} \frac{\rho\left(\mathbf{r}',t\right)}{\mathscr{r}}\,\mathrm{d}\tau'\,. \end{aligned} \tag{41}$$

Equation (41) is simply equation (19)! Therefore, V_1 satisfies Poisson's Equation and now we have that

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}. \tag{42}$$

V_2 is a bit harder to work with. Firstly, notice that taking the Laplacian in the unprimed coordinates is the same thing as taking the Laplacian in the "difference" coordinates. After all, if we let $(\tilde{x},\tilde{y},\tilde{z}) = (x-x', y-y', z-z') = \boldsymbol{\mathscr{r}}$, we have that, for each fixed $\mathbf{r}'$,

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right),\\ &= \frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\right),\\ &= \frac{\partial}{\partial\tilde{x}}\left(\frac{\partial}{\partial\tilde{x}}\right),\\ &= \frac{\partial^2}{\partial\tilde{x}^2}. \end{aligned} \tag{43}$$

With analogous calculations for y and z, we conclude that

$$\nabla^2 = \tilde{\nabla}^2. \tag{44}$$

This result was actually expected, since - for each fixed $\mathbf{r}'$ - $\boldsymbol{\mathscr{r}}$ is nothing but a translation of $\mathbf{r}$.

We are going to need to calculate the Laplacian of V_2 in spherical coordinates, which is given by

$$\begin{aligned} \nabla^2\xi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\xi}{\partial r}\right)\\ &\quad+ \frac{1}{r^2\sin\theta}\frac{\partial\xi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}, \end{aligned} \tag{45}$$

where we let r denote the distance from the origin, θ denotes the angle the position vector makes with the z-axis and ϕ denotes the angle that the position vector's projection onto the xy-plane makes with the x-axis. Luckily, we are physicists, and symmetry works on our behalf, which essentialy means we won't need to use equation (45) in all of its glory.

For each fixed point $\mathbf{r}'$, the distance to that point, $\mathscr{r} = \|\mathbf{r} - \mathbf{r}'\|$, is spherically symmetric with respect to $\mathbf{r}$. Accordingly, since

$$\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}$$

depends on $\mathbf{r}$ only through $\mathscr{r}$, it also has to be spherically symmetric. As an example, we know that the potential of a point charge at rest (which clearly has $\mathbf{r}'$) is spherically symmetric, for it depends only on the distance to the charge. Therefore, we might take the Laplacian in the $\mathscr{r}$ coordinates ignoring the last two terms of equation (45).

With the aid of equation (44), we can calculate the Laplacian of V_2 in order to find that

$$\begin{aligned}
\nabla^2 V_2 &= \nabla^2\left(\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\right),\\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\nabla^2\left(\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\left(\mathscr{r}^2\frac{\partial}{\partial\mathscr{r}}\left(\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}\right)\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\left(\mathscr{r}\frac{\partial}{\partial\mathscr{r}}\left(\rho\left(\mathbf{r}', t_r\right)\right)\right)\mathrm{d}\tau'\\
&\quad - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}}\frac{\partial^2}{\partial\mathscr{r}^2}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\\
&\quad + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\\
&\quad - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}}\frac{\partial^2}{\partial\mathscr{r}^2}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\,. \qquad (46)
\end{aligned}$$

This expression doesn't seem to be very nice yet. However, we know that a function of the form $u\left(t - \frac{x}{v}\right)$ satisfies the one-dimensional homogeneous wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0. \qquad (47)$$

Luckily, $\rho(t - \frac{\mathscr{r}}{c})$ is such a function! Therefore, it is true that

$$\frac{\partial^2}{\partial\mathscr{r}^2}\left(\rho\left(\mathbf{r}', t_r\right)\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\rho\left(\mathbf{r}', t_r\right)\right). \qquad (48)$$

By substituting equation (48) into equation (46), it follows that

$$\begin{aligned}
\nabla^2 V_2 &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{1}{\mathscr{r}}\cdot\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\rho\left(\mathbf{r}', t_r\right)\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}_2}\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\right). \qquad (49)
\end{aligned}$$

If now we let[9] $\mathcal{V}_1 \to 0$, and therefore $\mathcal{V}_2 \to \mathcal{V}$, equation (49) will read

$$\begin{aligned}
\nabla^2 V_2 &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}}\,\mathrm{d}\tau'\right),\\
&= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}. \qquad (50)
\end{aligned}$$

At last, we can sum equations (42) and (50) to find that

$$\begin{aligned}
\nabla^2 V_1 + \nabla^2 V_2 &= -\frac{\rho}{\epsilon_0} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},\\
\nabla^2(V_1 + V_2) - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0},\\
\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0}. \qquad (37)
\end{aligned}$$

Therefore, our guess did work for the scalar potential after all. The proof for the vector potential follows the same steps that we just did (after all, it is just the same equation on three different coordinates).

Although we can already find the electromagnetic fields in any problem we tackle (since we know how to find the potentials and equations (25) and (26) are the remaining step), we still haven't found what we are looking for: expressions for the physical fields themselves in terms of charges and currents, *i.e.*, improvements on Coulomb and Biot-Savart laws which still work when charges and currents are not static (and, of course, return to their previous versions when time is irrelevant). Thus, the next section starts the end of our journey.

However, I can't help to make a final remark about our work with retarded potentials: Riemann's argument would work equally well if we had picked $t_a = t + \frac{\mathscr{r}}{c}$ instead of t_r, which would mean that the potentials would be defined by the charge configurations in the future. Unfortunately, physicists tend to really like causality, which leads us to discard this solution[1].

VI. WHERE PHYSICAL INTUITION CAN'T GO...

One might now think "if simply calculating the integrands in the past worked fine for the potentials, it should work as well for the physical fields!". Actually, it is not that simple. Let's suppose that

---

[9] One should now not think about $\mathcal{V}_1$ as a set, but as its volume instead. We could also take the limit of the set in a rigorous way, for example through a sequence of decreasing sets which always include $\mathbf{r}$, but this level of enlightenment is not within the purposes of this work.

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\cdot\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'\,, \\ \mathbf{B}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\mathbf{J}(\mathbf{r}',t_r)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau'\,. \end{cases} \tag{51}$$

Under this pair of equations, what are we describing? Do these fields respect Maxwell's Equations? With some algebra about retarded quantitites[6], it is possible to show that the differential equations that rule these expressions are not Maxwell's and, therefore, our problem isn't solved. However, finding such expressions will tell us the problems in our suppositions, help us find the correct fields and give new intuition of the meaning of our final results.

Just before we start, we will write $\lfloor F \rfloor$ for a function F which is being evaluated at the retarded time (the dependence on $\mathbf{r}$ or $\mathbf{r}'$ should be understandable through the context).

For the Gauss Law, we have that

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\lfloor\rho\rfloor}{\mathscr{r}^2}\cdot\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'\,, \\ \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \boldsymbol{\nabla}\cdot\left(\frac{\lfloor\rho\rfloor}{\mathscr{r}^2}\cdot\hat{\boldsymbol{\mathscr{r}}}\right)\mathrm{d}\tau'\,, \\ &= \frac{1}{4\pi\epsilon_0}\int \lfloor\rho\rfloor\,\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) + \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\boldsymbol{\nabla}\lfloor\rho\rfloor\,\mathrm{d}\tau'\,, \\ &= \frac{1}{4\pi\epsilon_0}\int \lfloor\rho\rfloor\left(4\pi\delta^3\left(\boldsymbol{\mathscr{r}}\right)\right) + \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\boldsymbol{\nabla}\lfloor\rho\rfloor\,\mathrm{d}\tau'\,, \\ &= \frac{\rho}{\epsilon_0} + \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\boldsymbol{\nabla}\lfloor\rho\rfloor\,\mathrm{d}\tau'\,. \end{aligned} \tag{52}$$

Once more, δ^3 is the three-dimensional Dirac delta "function".

The gradient of $\lfloor\rho\rfloor$ isn't that helpful when we talk about keeping the calculation flow. To deal with it, and with many other retarded identities, it will be useful to think of them as[6]

$$\lfloor F \rfloor \equiv \int \delta\left(u\right) F(t')\,\mathrm{d}t'\,, \tag{53}$$

where $u := t' - t + \frac{\mathscr{r}}{c}$ and δ is the one-dimensional Dirac delta. With this idea, we can see that[10]

$$\begin{aligned} \boldsymbol{\nabla}\lfloor\rho\rfloor &= \int \boldsymbol{\nabla}(\delta\left(u\right))\rho\left(\mathbf{r}',t'\right)\mathrm{d}t'\,, \\ &= -\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial\delta\left(u\right)}{\partial t}\rho\left(\mathbf{r}',t'\right)\mathrm{d}t'\,, \\ &= -\frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial}{\partial t}\int \delta\left(u\right)\rho\left(\mathbf{r}',t'\right)\mathrm{d}t'\,, \\ &= -\frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial\lfloor\rho\rfloor}{\partial t}. \end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} - \frac{1}{4\pi\epsilon_0 c}\int \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\frac{\partial\lfloor\rho\rfloor}{\partial t}\,\mathrm{d}\tau'\,, \\ &= \frac{\rho}{\epsilon_0} - \frac{1}{4\pi\epsilon_0 c}\int \frac{1}{\mathscr{r}^2}\frac{\partial\lfloor\rho\rfloor}{\partial t}\,\mathrm{d}\tau'\,. \end{aligned} \tag{55}$$

By invoking equation (53) once more, we can see that

$$\begin{aligned} \frac{\partial\lfloor\rho\rfloor}{\partial t} &= \int \frac{\partial\delta\left(u\right)}{\partial t}\rho\left(\mathbf{r}',t'\right)\mathrm{d}t'\,, \\ &= -\int \frac{\partial\delta\left(u\right)}{\partial t'}\rho\left(\mathbf{r}',t'\right)\mathrm{d}t'\,, \\ &= \int \delta\left(u\right)\frac{\partial\rho\left(\mathbf{r}',t'\right)}{\partial t'} - \frac{\partial}{\partial t'}\left(\delta\left(u\right)\rho\left(\mathbf{r}',t'\right)\right)\mathrm{d}t'\,, \\ &= \left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \delta\left(u\right)\rho\left(\mathbf{r}',t'\right)\Big|_{-\infty}^{+\infty}\,, \\ &= \left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor. \end{aligned} \tag{56}$$

Finally, we get that

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{1}{4\pi\epsilon_0 c}\int \frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor\mathrm{d}\tau'\,. \tag{57}$$

Equation (57) clearly disagrees with Gauss's Law, which means that the electric field is definitely wrong. Nevertheless, let's continue the calculation of the other equations, so it is possible to see the effects of this wrong guess, which may give us a hint on what effects are not being taken into consideration.

Next, let's see what happens to Faraday's Law. By taking the curl of the $\mathbf{E}$ field defined in equation (51),

[10] To find the relations between the gradient and time derivatives of $\delta\left(u\right)$, one might use the chain rule to write the different derivatives in terms of u.

it follows that

$$
\begin{aligned}
\nabla \times \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \nabla \times \left(\frac{\lfloor\rho\rfloor}{\textit{ᴤ}^2}\cdot\hat{\textit{ᴤ}}\right)\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int \lfloor\rho\rfloor\,\nabla\times\left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\right) - \frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\times\nabla\lfloor\rho\rfloor\,\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int \lfloor\rho\rfloor\cdot\mathbf{0} + \nabla\lfloor\rho\rfloor\times\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,,\\
&= \frac{1}{4\pi\epsilon_0}\int \left(-\frac{\hat{\textit{ᴤ}}}{c}\frac{\partial\lfloor\rho\rfloor}{\partial t}\right)\times\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,,\\
&= -\frac{1}{4\pi\epsilon_0 c}\int \frac{1}{\textit{ᴤ}^2}\frac{\partial\lfloor\rho\rfloor}{\partial t}\left(\hat{\textit{ᴤ}}\times\hat{\textit{ᴤ}}\right)\mathrm{d}\tau'\,,\\
&= -\frac{1}{4\pi\epsilon_0 c}\int \mathbf{0}\,\mathrm{d}\tau'\,,\\
&= \mathbf{0}.
\end{aligned}
\tag{58}
$$

In the development of equation (58), we used equation (54) again.

Take a closer look at equation (58): by assuming these fields, we turned off electromagnetic induction of electric fields by the variation of magnetic fields.

To check the Magnetic Gauss's Law, we can simply take the divergence on both sides of the equation for $\mathbf{B}$ in equation (51):

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\int\frac{\lfloor\mathbf{J}\rfloor\times\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,,\\
\nabla\cdot\mathbf{B} &= \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\lfloor\mathbf{J}\rfloor\times\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\right)\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\left(\nabla\times\lfloor\mathbf{J}\rfloor\right)\\
&\quad + \lfloor\mathbf{J}\rfloor\cdot\left(\nabla\times\left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\right)\right)\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\left(\nabla\times\lfloor\mathbf{J}\rfloor\right) + \lfloor\mathbf{J}\rfloor\cdot\mathbf{0}\,\mathrm{d}\tau'\,.
\end{aligned}
\tag{59}
$$

Another use of equation (53) will yield that[11]

$$
\begin{aligned}
\nabla\times\lfloor\mathbf{J}\rfloor &= \int\nabla\times\left[\delta(u)\,\mathbf{J}(\mathbf{r}',t')\right]\mathrm{d}t'\,,\\
&= \int\delta(u)\,\nabla\times\mathbf{J} - \mathbf{J}\times\nabla\delta(u)\,\mathrm{d}t'\,,\\
&= \int\mathbf{0} + \nabla\delta(u)\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t'\,,\\
&= -\int\frac{\hat{\textit{ᴤ}}}{c}\frac{\partial\delta(u)}{\partial t}\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t'\,,\\
&= -\frac{\hat{\textit{ᴤ}}}{c}\times\frac{\partial}{\partial t}\int\delta(u)\,\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t'\,,\\
&= -\frac{\hat{\textit{ᴤ}}}{c}\times\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t},\\
&= \left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\times\frac{\hat{\textit{ᴤ}}}{c}.
\end{aligned}
\tag{60}
$$

The fact that $\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t} = \left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor$ can be proved following the same steps taken when proving equation (56). Substituting equation (60) into equation (59) we get that

$$
\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\left(\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\times\frac{\hat{\textit{ᴤ}}}{c}\right)\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\frac{1}{\textit{ᴤ}^2 c}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\cdot\left(\hat{\textit{ᴤ}}\times\hat{\textit{ᴤ}}\right)\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\mathbf{0}\,\mathrm{d}\tau'\,,\\
&= \mathbf{0}.
\end{aligned}
\tag{61}
$$

Finally, taking the curl of $\mathbf{B}$ will give us an analog for the Ampère-Maxwell Law.

$$
\begin{aligned}
\nabla\times\mathbf{B} &= \frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\lfloor\mathbf{J}\rfloor\times\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\right)\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\lfloor\mathbf{J}\rfloor\left(\nabla\cdot\left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\right)\right) + \left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\nabla\right)\lfloor\mathbf{J}\rfloor\,\mathrm{d}\tau'\\
&\quad - \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\nabla\cdot\lfloor\mathbf{J}\rfloor + \left(\lfloor\mathbf{J}\rfloor\cdot\nabla\right)\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,,\\
&= \frac{\mu_0}{4\pi}\int\lfloor\mathbf{J}\rfloor\left(4\pi\delta^3(\textit{ᴤ})\right) + \left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\nabla\right)\lfloor\mathbf{J}\rfloor\,\mathrm{d}\tau'\\
&\quad - \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\nabla\cdot\lfloor\mathbf{J}\rfloor + \left(\lfloor\mathbf{J}\rfloor\cdot\nabla\right)\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,,\\
&= \mu_0\mathbf{J} + \frac{\mu_0}{4\pi}\int\left(\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\cdot\nabla\right)\lfloor\mathbf{J}\rfloor\,\mathrm{d}\tau'\\
&\quad - \frac{\mu_0}{4\pi}\int\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\nabla\cdot\lfloor\mathbf{J}\rfloor + \left(\lfloor\mathbf{J}\rfloor\cdot\nabla\right)\frac{\hat{\textit{ᴤ}}}{\textit{ᴤ}^2}\,\mathrm{d}\tau'\,.
\end{aligned}
\tag{62}
$$

[11] It might be useful to note that, since $\mathbf{J}(\mathbf{r}',t')$ depends only on the primed coordinates, any of its derivatives with respect to the unprimed coordinates has to be zero.

If we want to proceed with the development of equation (62), it is going to be necessary to study the integrand on the RHS for a while. Firstly, notice that, since $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, it follows from the chain rule that

$$-\left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}'\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{63}$$

Consider then only the $\mathbf{x}$-component of the quantity in equation (63)[12]:

$$\begin{aligned}
\left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}'\right)\left(\frac{x-x'}{\mathscr{r}^3}\right) &= \left(\sum_{i=1}^{3}\lfloor J_{e_i}\rfloor\frac{\partial}{\partial e_i'}\right)\left(\frac{x-x'}{\mathscr{r}^3}\right),\\
&= \sum_{i=1}^{3}\left[\lfloor J_{e_i}\rfloor\frac{\partial}{\partial e_i'}\left(\frac{x-x'}{\mathscr{r}^3}\right)\right],\\
&= \lfloor\mathbf{J}\rfloor\cdot\left[\boldsymbol{\nabla}'\left(\frac{x-x'}{\mathscr{r}^3}\right)\right],\\
&= \boldsymbol{\nabla}'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\lfloor\mathbf{J}\rfloor\right)\\
&\quad-\left(\frac{x-x'}{\mathscr{r}^3}\right)\boldsymbol{\nabla}'\cdot\lfloor\mathbf{J}\rfloor.
\end{aligned} \tag{64}$$

However, notice that

$$\begin{aligned}
\boldsymbol{\nabla}'\cdot\lfloor\mathbf{J}\rfloor &= \int\boldsymbol{\nabla}'\cdot\left(\delta\left(u\right)\mathbf{J}\left(\mathbf{r}',t'\right)\right)\mathrm{d}t',\\
&= \int\boldsymbol{\nabla}'\delta\left(u\right)\cdot\mathbf{J}\left(\mathbf{r}',t'\right)\mathrm{d}t'\\
&\quad+\int\delta\left(u\right)\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{r}',t'\right)\mathrm{d}t',\\
&= -\int\boldsymbol{\nabla}\delta\left(u\right)\cdot\mathbf{J}\left(\mathbf{r}',t'\right)\mathrm{d}t' + \lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor,\\
&= -\int\boldsymbol{\nabla}\cdot\left(\delta\left(u\right)\mathbf{J}\left(\mathbf{r}',t'\right)\right)\mathrm{d}t' + \lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor,\\
&= -\boldsymbol{\nabla}\cdot\left(\int\delta\left(u\right)\mathbf{J}\left(\mathbf{r}',t'\right)\mathrm{d}t'\right) + \lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor,\\
&= -\boldsymbol{\nabla}\cdot\lfloor\mathbf{J}\rfloor + \lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor.
\end{aligned} \tag{65}$$

With equation (65) in our hands, equation (64) reads

$$\begin{aligned}
\left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}'\right)\left(\frac{x-x'}{\mathscr{r}^3}\right) &= \boldsymbol{\nabla}'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\lfloor\mathbf{J}\rfloor\right)\\
&\quad+\left(\frac{x-x'}{\mathscr{r}^3}\right)\boldsymbol{\nabla}\cdot\lfloor\mathbf{J}\rfloor\\
&\quad-\left(\frac{x-x'}{\mathscr{r}^3}\right)\lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor.
\end{aligned} \tag{66}$$

[12] To avoid big and repetitive equations, we shall write $\mathbf{x} = \mathbf{e_1}$, $\mathbf{y} = \mathbf{e_2}$ and $\mathbf{z} = \mathbf{e_3}$ at the following calculation and at some other computations through the text

Next, we are going to need to use the continuity equation:

$$\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0. \tag{67}$$

Since equation (67) is valid at all times and points, we are allowed to evaluate it in the sources on the retarded time in order to get[6]

$$\lfloor\boldsymbol{\nabla}'\cdot\mathbf{J}\rfloor + \left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor = 0. \tag{68}$$

By substituting equation (68) into equation (66) we obtain

$$\begin{aligned}
\left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}'\right)\left(\frac{x-x'}{\mathscr{r}^3}\right) &= \boldsymbol{\nabla}'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\lfloor\mathbf{J}\rfloor\right)\\
&\quad+\left(\frac{x-x'}{\mathscr{r}^3}\right)\boldsymbol{\nabla}\cdot\lfloor\mathbf{J}\rfloor\\
&\quad+\left(\frac{x-x'}{\mathscr{r}^3}\right)\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor.
\end{aligned} \tag{69}$$

Combining the respective equations for each of the coordinates, equation (69) implies that

$$\begin{aligned}
\left(\lfloor\mathbf{J}\rfloor\cdot\boldsymbol{\nabla}'\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} &= \sum_{i=1}^{3}\boldsymbol{\nabla}'\cdot\left(\frac{e_i-e_i'}{\mathscr{r}^3}\lfloor\mathbf{J}\rfloor\right)\hat{\mathbf{e}}_{\mathbf{i}}\\
&\quad+\sum_{i=1}^{3}\left[\left(\frac{e_i-e_i'}{\mathscr{r}^3}\right)\boldsymbol{\nabla}\cdot\lfloor\mathbf{J}\rfloor\right]\hat{\mathbf{e}}_{\mathbf{i}}\\
&\quad+\sum_{i=1}^{3}\left(\frac{e_i-e_i'}{\mathscr{r}^3}\right)\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor\hat{\mathbf{e}}_{\mathbf{i}},\\
&= \sum_{i=1}^{3}\boldsymbol{\nabla}'\cdot\left(\frac{e_i-e_i'}{\mathscr{r}^3}\lfloor\mathbf{J}\rfloor\right)\hat{\mathbf{e}}_{\mathbf{i}}\\
&\quad+\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\boldsymbol{\nabla}\cdot\lfloor\mathbf{J}\rfloor + \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor.
\end{aligned} \tag{70}$$

By substituting equations (63) and (70) into (62) we get that

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{\mu_0}{4\pi} \int \sum_{i=1}^{3} \boldsymbol{\nabla}' \cdot \left(\frac{e_i - e_i'}{\scriptr^3} \left\lfloor \mathbf{J} \right\rfloor \right) \hat{\mathbf{e}}_\mathbf{i} \, \mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \right) \left\lfloor \frac{\partial \rho}{\partial t} \right\rfloor \mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor \mathrm{d}\tau' , \\
&= \mu_0 \mathbf{J} + \frac{\mu_0}{4\pi} \sum_{i=1}^{3} \int_{\mathcal{V}} \boldsymbol{\nabla}' \cdot \left(\frac{e_i - e_i'}{\scriptr^3} \left\lfloor \mathbf{J} \right\rfloor \right) \mathrm{d}\tau' \, \hat{\mathbf{e}}_\mathbf{i} \\
&\quad + \frac{\mu_0}{4\pi} \int \frac{\hat{\scriptr}}{\scriptr^2} \frac{\partial \left\lfloor \rho \right\rfloor}{\partial t} \mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor \mathrm{d}\tau' , \\
&= \mu_0 \mathbf{J} + \frac{\mu_0}{4\pi} \sum_{i=1}^{3} \oint_{\partial \mathcal{V}} \frac{e_i - e_i'}{\scriptr^3} \left\lfloor \mathbf{J} \right\rfloor \cdot \mathrm{d}\mathbf{S}' \, \hat{\mathbf{e}}_\mathbf{i} \\
&\quad + \frac{\mu_0}{4\pi} \frac{\epsilon_0}{\epsilon_0} \frac{\partial}{\partial t} \int \frac{\left\lfloor \rho \right\rfloor}{\scriptr^2} \cdot \hat{\scriptr} \, \mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor \mathrm{d}\tau' , \\
&= \mu_0 \mathbf{J} + \frac{\mu_0}{4\pi} \sum_{i=1}^{3} 0 \cdot \hat{\mathbf{e}}_\mathbf{i} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\mathbf{E} \right) \\
&\quad + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor \mathrm{d}\tau' , \\
&= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\mu_0}{4\pi} \int \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor \mathrm{d}\tau' .
\end{aligned} \tag{71}
$$

In the previous calculation, the surface integrals vanished for the volume integrals were meant to be calculated over all space and it is physically necessary for $\lfloor \mathbf{J} \rfloor$ to be zero at infinite distance.

In order to deal with the remaining integrand in equation (71), we must once more write a retarded quantity as an integral. Notice that

$$
\begin{aligned}
\left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor &= \left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \int \delta\left(u\right) \mathbf{J}\left(\mathbf{r}', t'\right) \mathrm{d}t' , \\
&= \int \left[\left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \delta\left(u\right) \right] \mathbf{J}\left(\mathbf{r}', t'\right) \mathrm{d}t' .
\end{aligned} \tag{72}
$$

However, if we consider an identity we have been using concerning $\delta(u)$, we see that

$$
\begin{aligned}
\boldsymbol{\nabla} \delta\left(u\right) &= -\frac{\hat{\scriptr}}{c} \frac{\partial \delta\left(u\right)}{\partial t} , \\
\frac{\partial \delta\left(u\right)}{\partial x} &= -\frac{x - x'}{\scriptr c} \frac{\partial \delta\left(u\right)}{\partial t} , \\
\frac{x - x'}{\scriptr^3} \cdot \frac{\partial \delta\left(u\right)}{\partial x} &= -\frac{\left(x - x'\right)^2}{\scriptr^4 c} \frac{\partial \delta\left(u\right)}{\partial t} , \\
\left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \delta\left(u\right) &= -\frac{\scriptr^2}{\scriptr^4 c} \frac{\partial \delta\left(u\right)}{\partial t} , \\
\left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \delta\left(u\right) &= -\frac{1}{\scriptr^2 c} \frac{\partial \delta\left(u\right)}{\partial t} .
\end{aligned} \tag{73}
$$

Thus,

$$
\begin{aligned}
\left(\frac{\hat{\scriptr}}{\scriptr^2} \cdot \boldsymbol{\nabla} \right) \left\lfloor \mathbf{J} \right\rfloor &= \int -\frac{1}{\scriptr^2 c} \frac{\partial \delta\left(u\right)}{\partial t} \mathbf{J}\left(\mathbf{r}', t'\right) \mathrm{d}t' , \\
&= -\frac{1}{\scriptr^2 c} \frac{\partial}{\partial t} \int \delta\left(u\right) \mathbf{J}\left(\mathbf{r}', t'\right) \mathrm{d}t' , \\
&= -\frac{1}{\scriptr^2 c} \frac{\partial \left\lfloor \mathbf{J} \right\rfloor}{\partial t} .
\end{aligned} \tag{74}
$$

Substituting equation (74) into equation (71) will yield

$$
\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \frac{\mu_0}{4\pi c} \int \frac{1}{\scriptr^2} \left\lfloor \frac{\partial \mathbf{J}}{\partial t} \right\rfloor \mathrm{d}\tau' . \tag{75}
$$

Bringing equations (57), (58), (61) and (75) together, we se that the fields on equation (51) are such that

$$
\begin{cases}
\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} - \dfrac{1}{4\pi\epsilon_0 c} \displaystyle\int \frac{1}{\scriptr^2} \left\lfloor \frac{\partial \rho}{\partial t} \right\rfloor \mathrm{d}\tau' , \\
\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \\
\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}, \\
\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} - \dfrac{\mu_0}{4\pi c} \displaystyle\int \frac{1}{\scriptr^2} \left\lfloor \frac{\partial \mathbf{J}}{\partial t} \right\rfloor \mathrm{d}\tau' .
\end{cases} \tag{76}
$$

Equation (76) shows clearly that equation (51) aren't the solutions for Maxwell's Equations. On the bright side, we got $\boldsymbol{\nabla} \cdot \mathbf{B}$ correctly!

Equation (76) actually fail even to describe conservation of charge! By taking the divergence of our not-quite-Ampère-Maxwell Law, it is seen that

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) &= \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} + \mu_0 \epsilon_0 \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) \\
&\quad - \frac{\mu_0}{4\pi c} \boldsymbol{\nabla} \cdot \left(\int \frac{1}{\scriptr^2} \left\lfloor \frac{\partial \mathbf{J}}{\partial t} \right\rfloor \mathrm{d}\tau' \right) .
\end{aligned} \tag{77}
$$

Since the curl of a vector field is always divergenceless and equation (56) applies to any function just as it does to ρ, it follows that

$$\begin{aligned}
0 &= \nabla\cdot\mathbf{J} + \epsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0} - \frac{1}{4\pi\epsilon_0 c}\int\frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor \mathrm{d}\tau'\right)\\
&\quad - \frac{1}{4\pi c}\int\nabla\cdot\left(\frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\right)\mathrm{d}\tau'\,,\\
&= \nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} - \frac{1}{4\pi c}\int\frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial^2\rho}{\partial t^2}\right\rfloor\mathrm{d}\tau'\\
&\quad - \frac{1}{4\pi c}\int\nabla\cdot\left(\frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\right)\mathrm{d}\tau'\,,\\
&= \nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} - \frac{1}{4\pi c}\int\frac{1}{\mathscr{r}^2}\left\lfloor\frac{\partial^2\rho}{\partial t^2}\right\rfloor\mathrm{d}\tau'\\
&\quad - \frac{1}{4\pi c}\int\frac{1}{\mathscr{r}^2}\nabla\cdot\left(\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\right)\mathrm{d}\tau'\\
&\quad - \frac{1}{4\pi c}\int\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\cdot\nabla\left(\frac{1}{\mathscr{r}^2}\right)\mathrm{d}\tau'\,,\\
&= \nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}\\
&\quad - \frac{1}{4\pi c}\int\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial t}\left(\frac{\partial\lfloor\rho\rfloor}{\partial t} + \nabla\cdot\lfloor\mathbf{J}\rfloor\right)\mathrm{d}\tau' \qquad (78)\\
&\quad + \frac{1}{4\pi c}\int\frac{2}{\mathscr{r}^3}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\cdot\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'\,.
\end{aligned}$$

Equation (78) looks incredibly different from the Continuity Equation (which, in order to obtain $\nabla\times\mathbf{B}$, we actually imposed on equation (67) as a basic principle!), but that is in fact consistent with electromagnetic theory: if the Continuity Equation was satisfied, then so would Maxwell's Equations[6], and we have already seen that these fields can be anything but the solutions we seek.

Since physical intuition seems to have abandoned us, what can we do next? Besides, what did we do wrong?

Firstly, notice that we ignored the fact that currents and charges are essentially the same physical objects at different speeds and, therefore, are mathematically related (namely, by the Continuity Equation). We can, thus, expect for $\mathbf{E}$ to depend on $\mathbf{J}$ or for $\mathbf{B}$ to depend on ρ (or both).

Equations (76) and (78) give us some hints on this. The "Uncontinuity" Equation suggests that we have not treated the relations between ρ and $\mathbf{J}$ properly. The failure on Faraday's Law suggests the same issue: our description says that $\mathbf{E}$ depends only on ρ and $\mathbf{B}$ depends only on $\mathbf{J}$. Therefore, if there are issues concerning how $\mathbf{E}$ and $\mathbf{B}$ are related, these issues can be traced back to our lack of respect with how we should treat ρ and $\mathbf{J}$.

Furthermore, our guess never bothered at all with how fast the quantities involved are changing, what might be curious, since the whole idea rests on the idea that the fields depend upon the charges and currents after some time has elapsed. Are we sure that there is no dependence on the temporal derivatives, for example?

Equation (76) suggests this as well. After all, Gauss's Law and Ampère-Maxwell Law received new terms depending on the time-derivatives of the charges and currents. This gives us the hint that if such quantities were present in the initial fields on a certain way, their divergences and curls would not have these problems.

A final argument comes from equations (26) and (38): $\mathbf{E}$ depends on the time-derivatives of $\mathbf{A}$, which depends on $\mathbf{J}$. It is strongly expected that $\mathbf{E}$ not only depends on $\mathbf{J}$, but on its time-derivative.

How can we take these problems into account when making a new approach?

VII. ...WE BRING MATH

Apart from the mathematical hints we got from equations (76) and (78), the same arguments could be applied in order to criticize our approach towards the retarded potentials. Therefore, why did that guess did work?

The answer is simple: we did not rely only on physical intuition, but also on the equations we had at hand. Although me might think that the time derivatives might influence the retarded potentials or that there could be a current dependence on the scalar potential (for an example), we had a wave equation at hand (equation (37)) to support our guess. It was the mathematical structure of the differential equations that described the potentials suggested us that solution, not simply the physical considerations we could make on the problem. Thus, lets retrace our procedures in order to find the correct expressions for the physical fields.

Our derivation of the retarded potentials (equation (38)) started by taking a look at the wave equations they respected (equation (37)), identifying their sources and then solving the equations[13]. However, in our previous attempt to find the equations for the physical fields we ignored the labor of finding a wave equation and simply guessed that there should be one with ρ and $\mathbf{J}$ as sources! In order to fix this, let's take our steps once more.

[13] It might seem odd to say that we solved a partial differential equation without specifying the boundary and initial conditions. Implicitly, we assumed the fields must vanish at infinity, when they are apart from any charges and currents. We also hid the homogenous component of the solution, which can be understood as fields which were not originated by charges or currents, but were present in the Universe from the very beginning. Finally, the temporal dependence hides in the expressions for the charges and currents as well. Essentially, we are studying the forced oscillations caused in the electromagnetic fields by charges and currents without taking into account the free oscillations. I appreciate João C. A. Barata's precious explanations when looking for the initial condition's hideout.

Let's start with Faraday's Law and take the curl of each side of the equation. We will then have that

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t},\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})&=-\boldsymbol{\nabla}\times\left(\frac{\partial\mathbf{B}}{\partial t}\right),\\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})-\nabla^2\mathbf{E}&=-\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\mathbf{B}\right).\end{aligned}\tag{79}$$

To keep it going, we will also need to remember Gauss's Law and Ampère-Maxwell Law, in order to get rid of $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{B}$.

$$\begin{aligned}\boldsymbol{\nabla}\left(\frac{\rho}{\epsilon_0}\right)-\nabla^2\mathbf{E}&=-\frac{\partial}{\partial t}\left(\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right),\\ \frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho-\nabla^2\mathbf{E}&=-\mu_0\frac{\partial\mathbf{J}}{\partial t}-\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2},\\ \nabla^2\mathbf{E}-\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}&=\frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho+\mu_0\frac{\partial\mathbf{J}}{\partial t}.\end{aligned}\tag{80}$$

Through a similar procedure, we can start with Ampère-Maxwell Law and see that

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B}&=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t},\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})&=\mu_0\boldsymbol{\nabla}\times\mathbf{J}+\mu_0\epsilon_0\boldsymbol{\nabla}\times\left(\frac{\partial\mathbf{E}}{\partial t}\right),\\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B})-\nabla^2\mathbf{B}&=\mu_0\boldsymbol{\nabla}\times\mathbf{J}+\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\mathbf{E}\right).\end{aligned}\tag{81}$$

If we recall Faraday's Law and the Magnetic Gauss's Law, it will follow that

$$\begin{aligned}\boldsymbol{\nabla}(0)-\nabla^2\mathbf{B}&=\mu_0\boldsymbol{\nabla}\times\mathbf{J}-\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{B}}{\partial t}\right),\\ -\nabla^2\mathbf{B}&=\mu_0\boldsymbol{\nabla}\times\mathbf{J}-\mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2},\\ \nabla^2\mathbf{B}-\mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}&=-\mu_0\boldsymbol{\nabla}\times\mathbf{J}.\end{aligned}\tag{82}$$

Writing equations (80) and (82) together we see that

$$\begin{cases}\nabla^2\mathbf{E}-\mu_0\epsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}=\dfrac{1}{\epsilon_0}\boldsymbol{\nabla}\rho+\mu_0\dfrac{\partial\mathbf{J}}{\partial t},\\ \nabla^2\mathbf{B}-\mu_0\epsilon_0\dfrac{\partial^2\mathbf{B}}{\partial t^2}=-\mu_0\boldsymbol{\nabla}\times\mathbf{J}.\end{cases}\tag{83}$$

We've just found out that the physical fields also obey a pair of wave equations! Furthermore, now we know which quantities actually represent the sources of these electromagnetic waves: not ρ and $\mathbf{J}$, as we thought before, but $\boldsymbol{\nabla}\rho$, $\frac{\partial\mathbf{J}}{\partial t}$ and $\boldsymbol{\nabla}\times\mathbf{J}$ instead. These differential equations agree with our previous expectation that $\mathbf{E}$ would depend on the time-derivative of $\mathbf{J}$.

The best part of finding these wave equations is that now we are facing a problem we have already solved before. Equation (83) are essentially the same as equation (37), and the solution to this equation was given by the retarded potentials in equation (38). Therefore, since the mathematical structures are essentially the same, the solutions to equation (83) must be given by

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=-\dfrac{1}{4\pi\epsilon_0}\displaystyle\int\frac{\lfloor\boldsymbol{\nabla}'\rho\rfloor}{\mathscr{r}}\,\mathrm{d}\tau'-\frac{\mu_0}{4\pi}\int\frac{1}{\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\mathrm{d}\tau'\\ \mathbf{B}(\mathbf{r},t)=\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\lfloor\boldsymbol{\nabla}'\times\mathbf{J}\rfloor}{\mathscr{r}}\,\mathrm{d}\tau'\end{cases}\tag{84}$$

The same argument given in Section V to prove the solution given for V applies in here for the physical fields we just proposed.

Although equation (84) does solve Maxwell's Equations, they look very different from equations (20) and (21)! We can't see yet if those equations are satisfied when we go back to static charges and currents, or if they are true under other conditions[7]! Therefore, our work in here isn't over yet and it is time to find a better expression.

Firstly, we already know an algorithm to deal with retarded quantities (namely, equation (53)). Let's then apply it to express $\lfloor\boldsymbol{\nabla}'\rho\rfloor$ in another way:

$$\begin{aligned}\lfloor\boldsymbol{\nabla}'\rho\rfloor&=\int\delta\left(u\right)\boldsymbol{\nabla}'\rho(\mathbf{r}',t')\,\mathrm{d}t',\\ &=\int\boldsymbol{\nabla}'\left(\delta\left(u\right)\rho(\mathbf{r}',t')\right)-\boldsymbol{\nabla}'\delta\left(u\right)\rho(\mathbf{r}',t')\,\mathrm{d}t',\\ &=\boldsymbol{\nabla}'\int\delta\left(u\right)\rho\,\mathrm{d}t'+\int\boldsymbol{\nabla}\delta\left(u\right)\rho\,\mathrm{d}t',\\ &=\boldsymbol{\nabla}'\lfloor\rho\rfloor-\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial\delta\left(u\right)}{\partial t}\rho(\mathbf{r}',t')\,\mathrm{d}t',\\ &=\boldsymbol{\nabla}'\lfloor\rho\rfloor-\frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial}{\partial t}\int\delta\left(u\right)\rho(\mathbf{r}',t')\,\mathrm{d}t',\\ &=\boldsymbol{\nabla}'\lfloor\rho\rfloor-\frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial\lfloor\rho\rfloor}{\partial t}.\end{aligned}\tag{85}$$

Joining equations (84) and (85), we see that

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\boldsymbol{\nabla}'\lfloor\rho\rfloor}{\mathscr{r}}\,\mathrm{d}\tau' \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' , \\
&= -\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \boldsymbol{\nabla}'\left(\frac{\lfloor\rho\rfloor}{\mathscr{r}}\right) - \lfloor\rho\rfloor\,\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau' \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' , \\
&= -\frac{1}{4\pi\epsilon_0}\oint_{\partial\mathcal{V}} \frac{\lfloor\rho\rfloor}{\mathscr{r}}\,\mathrm{d}\mathbf{S}' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \lfloor\rho\rfloor\,\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau' \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' , \\
&= \mathbf{0} + \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\lfloor\rho\rfloor + \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor \mathrm{d}\tau' \\
&\quad - \frac{1}{4\pi\epsilon_0}\int \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' , \\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\lfloor\rho\rfloor + \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' .
\end{aligned} \tag{86}$$

In order to obtain equation (86), we used the fact from vector calculus that[14]

$$\int_{\mathcal{V}} \boldsymbol{\nabla} T\,\mathrm{d}\tau = \oint_{\partial\mathcal{V}} T\,\mathrm{d}\mathbf{S} . \tag{87}$$

Besides, the surface integral vanished due to the fact that the volume integral was to be carried on the whole space and the charges vanish at infinity.

To "open" the integral for the magnetic field in equation (84), we need to understand $\lfloor\boldsymbol{\nabla}'\times\mathbf{J}\rfloor$. We can see that

$$\begin{aligned}
\lfloor\boldsymbol{\nabla}'\times\mathbf{J}\rfloor &= \int \delta(u)\,\boldsymbol{\nabla}'\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' , \\
&= \int \boldsymbol{\nabla}'\times\left(\delta(u)\,\mathbf{J}(\mathbf{r}',t')\right)\mathrm{d}t' \\
&\quad - \int \boldsymbol{\nabla}'\delta(u)\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' , \\
&= \boldsymbol{\nabla}'\times\int \delta(u)\,\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' \\
&\quad + \int \boldsymbol{\nabla}\delta(u)\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' , \\
&= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor - \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\frac{\partial\delta(u)}{\partial t}\times\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' , \\
&= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor - \frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\times\frac{\partial}{\partial t}\int \delta(u)\,\mathbf{J}(\mathbf{r}',t')\,\mathrm{d}t' , \\
&= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor - \frac{\hat{\boldsymbol{\mathscr{r}}}}{c}\times\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t} .
\end{aligned} \tag{88}$$

By substituting equation (88) into equation (84) we obtain that

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}\boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor - \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\times\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t}\,\mathrm{d}\tau' , \\
&= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}\boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor + \frac{1}{c\,\mathscr{r}}\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t}\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' , \\
&= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \boldsymbol{\nabla}'\times\left(\frac{\lfloor\mathbf{J}\rfloor}{\mathscr{r}}\right) - \boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\times\lfloor\mathbf{J}\rfloor\,\mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{c\,\mathscr{r}}\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t}\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' , \\
&= -\frac{\mu_0}{4\pi}\int_{\partial\mathcal{V}} \frac{\lfloor\mathbf{J}\rfloor}{\mathscr{r}}\times\mathrm{d}\mathbf{S}' - \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\times\lfloor\mathbf{J}\rfloor\,\mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{c\,\mathscr{r}}\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t}\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' , \\
&= \mathbf{0} + \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}^2}\lfloor\mathbf{J}\rfloor\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' \\
&\quad + \frac{\mu_0}{4\pi}\int \frac{1}{c\,\mathscr{r}}\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t}\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' , \\
&= \frac{\mu_0}{4\pi}\int \left[\frac{1}{\mathscr{r}^2}\lfloor\mathbf{J}\rfloor + \frac{1}{c\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\right]\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' .
\end{aligned} \tag{89}$$

Again, we used a fact known from vector calculus[15]:

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\times\mathbf{T}\,\mathrm{d}\tau = -\oint_{\partial\mathcal{V}} \mathbf{T}\times\mathrm{d}\mathbf{S} . \tag{90}$$

Just as before, the surface integral vanishes due to physical restraints: the currents must vanish at infinity.

Finally, equations (86) and (89) read

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\lfloor\rho\rfloor + \dfrac{\hat{\boldsymbol{\mathscr{r}}}}{c\mathscr{r}}\left\lfloor\dfrac{\partial\rho}{\partial t}\right\rfloor - \dfrac{1}{c^2\mathscr{r}}\left\lfloor\dfrac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau' , \\ \mathbf{B}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \left[\dfrac{1}{\mathscr{r}^2}\lfloor\mathbf{J}\rfloor + \dfrac{1}{c\mathscr{r}}\left\lfloor\dfrac{\partial\mathbf{J}}{\partial t}\right\rfloor\right] \times \hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau' . \end{cases} \tag{91}$$

Equation (91) are known as Jefimenko's Equations, which seem to have been first published in this form by Jefimenko[1, 8] in his textbook, [9]. However, Panofsky and Phillips found an equivalent expression in their textbook, [10]. Equation (14-34) is already identical to Jefimenko's expression for the magnetic field. The expression for the electric field is also equal to the one we found, and this can be seen by applying the Inverse Fourier Transform to equation (14-36).

Notice that Jefimenko's Equations are indeed general forms of Coulomb and Biot-Savart Laws. If we hold ρ and $\mathbf{J}$ constant as time goes by (*i.e.*, impose $\frac{\partial\rho}{\partial t} = 0$ and $\frac{\partial\mathbf{J}}{\partial t} = \mathbf{0}$), equation (91) recovers equations (20) and (21). Moreover, they exhibit the validity domain of Coloumb and Biot-Savart Laws (*e.g.*, we can see that Biot-Savart Law holds regardless of the time-dependence of ρ, as long as $\mathbf{J}$ is constant over time)[7].

As we expected, Jefimenko's Equations solves the issues we found with our guess at Section (VI): they do depend on time-derivatives and $\mathbf{E}$ does depends on $\mathbf{J}$ (actually, $\frac{\partial\mathbf{J}}{\partial t}$). It is interesting that the terms that once appeared at $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{B}$ now appear at $\mathbf{E}$ and $\mathbf{B}$, with few modifications. In addition, the time-derivative of $\mathbf{J}$ appearing at the expression for $\mathbf{E}$ shall fix the issue with Faraday's Law, as predicted (previously, we stated that ignoring the connection between ρ and $\mathbf{J}$ would lead to the lack of connection between $\mathbf{E}$ and $\mathbf{B}$).

The importance of these equations, besides the feeling of closure given to Classical Electrodynamics, relies on their physical meaning: we see now that instead of a cycling causality between the physical fields, with $\mathbf{B}$ generating $\mathbf{E}$ and vice versa, they are directly caused by the charges and currents. They also might be used to find the Heaviside-Feynman expressions for the fields generated by a point charge[11].

VIII. CONCLUSIONS

Although Jefimenko's Equations (equation (91)) may make the potential formulation look unnecessary, one should keep in mind that eventually he or she would need to actually compute all the integrals involved when trying to describe the motion of charges. It is, in general, easier to solve four problems (the potentials) instead of six (the physical fields) when applying the theory. Indeed, one could obtain Jefimenko's Equations by taking the derivatives (equations (25) and (26)) of the retarded potentials (equation (38))[1].

Nevertheless, Jefimenko's Equation still exhibit physical meanings that were previously hidden and are tools which can be used to tackle new problems. Under their light, we can see the independence between electric and magnetic fields: Faraday's Law is a consequence of the dependence of the electromagnetic fields on the charges and currents. Although changes in the magnetic field seem to cause changes in the electric field, this is much more a coincidence than it is a fact: the entities responsible for the changes in the magnetic field are also responsible for changes in the electric field, for each one of them can be determined independently of the other by simply knowing the charge and current distributions.

Finally, as noted by Lemos in [8], physical intuition should be handled with care, since it can easily lead us towards mistakes. The appropriate mathematical attention should be taken to avoid such confusions. As I read once in a while at the office of some dear friends,

> Where physical intuition can't go, we bring math; *i.e.* we bring math almost everywhere.

Unfortunately, the author of this phrase is still unknown to me.

ACKNOWLEDGEMENTS

I must express my gratitude towards my advisor, João Carlos Alves Barata, and my friend Victor Bernardo Chabu for their support and most valuable conversations concerning this paper. Their advice guided me through the paths that were beyond the reach of books and mere vector calculus.

Furthermore, I am thankful for my friends endless support, kindness and, specially, for their enriching comments on the seminar that first gave me the will to write this paper.

[14] This theorem might be a bit more obscure than his older brother, the Divergence Theorem. Therefore, Appendix contains a proof sketch.

[15] Just as for equation (87), you can find an argument in defense of this identity at the Appendix .

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Appendix: Corollaries of vector calculus

The main goal of this appendix is to "prove" equation (87),

$$\int_{\mathcal{V}} \boldsymbol{\nabla} T \,\mathrm{d}\tau = \oint_{\partial\mathcal{V}} T \,\mathrm{d}\mathbf{S}\,, \tag{87}$$

and equation (90),

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \times \mathbf{T} \,\mathrm{d}\tau = -\oint_{\partial\mathcal{V}} \mathbf{T} \times \mathrm{d}\mathbf{S}\,, \tag{90}$$

which were neccessary at Section VII.

In order to prove equation (87), let's start with the Divergence Theorem:

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{T} \,\mathrm{d}\tau = \oint_{\partial\mathcal{V}} \mathbf{T} \cdot \mathrm{d}\mathbf{S}\,. \tag{A.1}$$

Take $\mathbf{T}$ to be given by $\mathbf{T} = \mathbf{k}F$, where $\mathbf{k}$ is a constant vector. It follows that

$$\begin{aligned}
\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\mathbf{k}F) \,\mathrm{d}\tau &= \oint_{\partial\mathcal{V}} (\mathbf{k}F) \cdot \mathrm{d}\mathbf{S}\,,\\
\int_{\mathcal{V}} F \cdot \boldsymbol{\nabla} \cdot \mathbf{k} + \mathbf{k} \cdot \boldsymbol{\nabla} F \,\mathrm{d}\tau &= \mathbf{k} \cdot \oint_{\partial\mathcal{V}} F \,\mathrm{d}\mathbf{S}\,,\\
\int_{\mathcal{V}} \mathbf{k} \cdot \boldsymbol{\nabla} F \,\mathrm{d}\tau &= \mathbf{k} \cdot \oint_{\partial\mathcal{V}} F \,\mathrm{d}\mathbf{S}\,,\\
\mathbf{k} \cdot \int_{\mathcal{V}} \boldsymbol{\nabla} F \,\mathrm{d}\tau &= \mathbf{k} \cdot \oint_{\partial\mathcal{V}} F \,\mathrm{d}\mathbf{S}\,.
\end{aligned} \tag{A.2}$$

Since this expression holds for any constant vector $\mathbf{k}$, we conclude that

$$\int_{\mathcal{V}} \boldsymbol{\nabla} F \,\mathrm{d}\tau = \oint_{\partial\mathcal{V}} F \,\mathrm{d}\mathbf{S}\,, \tag{A.3}$$

which is equation (87).

A similar reasoning allows us to prove equation (90). Let $\mathbf{T} = \mathbf{F} \times \mathbf{k}$, where $\mathbf{k}$ is, once more, a constant vector. It follows that

$$\begin{aligned}
\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{k}) \,\mathrm{d}\tau &= \oint_{\partial\mathcal{V}} (\mathbf{F} \times \mathbf{k}) \cdot \mathrm{d}\mathbf{S}\,,\\
\int_{\mathcal{V}} \mathbf{k} \cdot \boldsymbol{\nabla} \times \mathbf{F} - \mathbf{F} \cdot \boldsymbol{\nabla} \times \mathbf{k} \,\mathrm{d}\tau &= -\oint_{\partial\mathcal{V}} (\mathbf{k} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}\,,\\
\int_{\mathcal{V}} \mathbf{k} \cdot \boldsymbol{\nabla} \times \mathbf{F} - 0 \,\mathrm{d}\tau &= -\oint_{\partial\mathcal{V}} \mathbf{k} \cdot (\mathbf{F} \times \mathrm{d}\mathbf{S}),\\
\mathbf{k} \cdot \int_{\mathcal{V}} \boldsymbol{\nabla} \times \mathbf{F} \,\mathrm{d}\tau &= -\mathbf{k} \cdot \oint_{\partial\mathcal{V}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}\,.
\end{aligned} \tag{A.4}$$

Since the expression holds for any constant vector $\mathbf{k}$, it follows that

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \times \mathbf{F} \,\mathrm{d}\tau = -\oint_{\partial\mathcal{V}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}\,. \tag{A.5}$$

This proves equation (90).