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An Introduction to Measure Theory

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Abstract

These are some study notes I've been developing while studying Measure Theory. Most of the material presented in here was strongly based upon Prof. Walter Pedra's course on Mathematical Physics III , [2], presented at the Institute of Physics at the University of São Paulo in Fall 2020. Indeed, these notes were originally written as a way to keep up with the course.

I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments and etc through my e-mail (nickolas@fma.if.usp.br). If you wish to have a look at more works, please check my personal website `http://fma.if.usp.br/~nickolas`.

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Contents

1 What is the size of a set?

The motivation for the study of Measure Theory comes from the notion of lengths, areas, and volumes. These concepts can be regarded as functions $m\colon \operatorname{Dom} m \to \bar{\mathbb{R}}$ with some specific properties, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$*. Usually, we know how to compute the lenghts, areas and volumes of lines, squares, cubes and even more complicated shapes, such as the ones found when dealing with multiple integrals in multidimensional Analysis. Our goal is to obtain a function that attributes the volume† of an arbitrary subset of $\mathbb{R}^n$. Thus, we pick $\operatorname{Dom} m = \mathbb{P}(\mathbb{R}^n)$.

First of all, we know volumes are strictly positive, and thus we may demand in our definition that $m(E) \geqslant 0, \forall\, E \in \mathbb{P}(\mathbb{R}^n)$.

We know that the volume of two disjoint sets if simply the sum of the volumes of each individual set. Therefore, $m(E \sqcup F) = m(E) + m(F), \forall\, E, F \in \mathbb{P}(\mathbb{R}^n)$.

Notation:

We denote the union of two disjoint sets E, F as $E \sqcup F$. The set $E \sqcup F$ is identical to the set $E \cup F$, but the notation $\sqcup$ indicates that $E \cap F = \varnothing$. ♦

We also know volumes are invariant under some kinds of transformation. If a transformation $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves the distance between points - *exempli gratia*, rotations and translations -, then is should hold that $m(\beta(E)) = m(E), \forall\, E \in \mathbb{P}(\mathbb{R}^n)$. We shall refer to these transformations β with a certain frequency, which motivates us to define a movement.

Definition 1 [Movement]**:**

Let $n \in \mathbb{N}$ and consider the metric space $(\mathbb{R}^n, d)$, where d is the Euclidean metric. A *movement* in $\mathbb{R}^n$ is an isometry $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$, *id est*, a bijection from $\mathbb{R}^n$ onto itself such that

$$d\left(\beta(x), \beta(y)\right) = d(x, y), \forall\, x, y \in \mathbb{R}^n. \tag{1.1}$$

♠

Notice that the movement of a set $E \in \mathbb{P}(\mathbb{R}^n)$ is simply the image of the set under that movement. Thus, we demand m to be invariant under movements.

*We shall make this more formal in a while. For now, let us concentrate on the intuitive notions.

†For simplicity, let us say only "volume", but of course the same could be said about lengths and areas

Finally, if we expect to obtain a uniqueness theorem concerning the existence of volumes on $\mathbb{R}^n$, we should impose a normalization condition. Therefore, we shall require that $m\left((0,1]^n\right) = 1$.

We now have four axioms concerning the function $m\colon \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}$:

i. $m(E) \geqslant 0, \forall\, E \in \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}$;

ii. $m(E \sqcup F) = m(E) + m(F), \forall\, E, F \in \mathbb{P}(\mathbb{R}^n)$;

iii. for any movement $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$, $(m \circ \beta)(E) = m(E), \forall\, E \in \mathbb{P}(\mathbb{R}^n)$;

iv. $m\left((0,1]^n\right) = 1$.

The problem we want to solve is, $\forall\, n \in \mathbb{N}$, to find such a function.

Sadly, it has been proven by Hausdorff in 1914 that such a problem has no solution in $\mathbb{R}^n$ with $n \geqslant 3$. In 1924, Banach showed the problem does admit a solution for $n < 3$, but such a solution is not unique.

The reason the problem fails to have a solution for $n \geqslant 3$ is the fact that ZFC (Zermelo-Frankel Set Theory with the Axiom of Choice) does not respect the following Galilean principle:

> A body, upon being separated in finitely many parts, can be recombined through movements in a way such that the final occupied volume exceeds the initial volume.

Indeed, the Axiom of Choice allows for the proof of the Banach-Tarski Theorem.

Banach-Tarski Theorem:

Let $n \in \mathbb{N}, n \geqslant 3$. *Let* $E, F \in \mathbb{P}(\mathbb{R}^n)$ *be limited sets with non-empty interior. Then there are* $k \in \mathbb{N}, E_i \in \mathbb{P}(E_i), i \in \{i\}_{i=1}^k; E_i \cap E_j = \varnothing, \forall\, i \neq j$ *and movements* $\beta_i\colon \mathbb{R}^n \to \mathbb{R}^n, i \in \{i\}_{i=1}^k$ *such that*

$$\bigsqcup_{i=1}^{k} E_i = E, \quad \bigcup_{i=1}^{k} \beta_i(E_i) = F. \tag{1.2}$$

□

If there were a function $m\colon \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}$ satisfying the properties we have previously required, this would lead to contradiction.

Firstly, notice that induction allows us to conclude that

$$m\left(\bigsqcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i). \tag{1.3}$$

This holds for any collection of finitely many disjoint sets.

Furthermore, notice that $E \subseteq F \Rightarrow m(E) \leqslant m(F)$. Indeed,

$$\begin{aligned} F &= (F \cap E) \sqcup (F \cap E^c), \\ F &= E \sqcup F \backslash E, \\ m(F) &= m(E) + m(F \backslash E), \\ m(F) &\geqslant m(E). \end{aligned} \tag{1.4}$$

For every $i \in \{i\}_{i=1}^{k}$, let us define $F_1 = E_1$ and $F_{i+1} = E_{i+1} \setminus \bigsqcup_{j=1}^{i} F_j$. This defines a finite sequence of sets such that $F_i \subseteq E_i, \forall \in \{i\}_{i=1}^{k}$ and $\bigsqcup_{i=1}^{k} F_i = \bigcup_{i=1}^{k} E_i$. Thus, we have

$$\begin{aligned} m\left(\bigcup_{i=1}^{k} E_i\right) &= m\left(\bigsqcup_{i=1}^{k} F_i\right), \\ &= \sum_{i=1}^{k} m(F_i), \\ &\leqslant \sum_{i=1}^{k} m(E_i). \end{aligned} \tag{1.5}$$

Therefore, if we consider the statement of the Banach-Tarski Theorem, we see that

$$m(E) = \sum_{i=1}^{k} m(E_i). \tag{1.6}$$

Furthermore, since m is invariant under movements,

$$\begin{aligned} m(F) &= m\left(\bigcup_{i=1}^{k} \beta_i(E_i)\right), \\ &\leqslant \sum_{i=1}^{k} m(\beta_i(E_i)), \\ &= \sum_{i=1}^{k} m(E_i), \\ &= m(E). \end{aligned} \tag{1.7}$$

This might not seen like a problem at first. However, notice that the result holds for any F. In particular, let us pick $E = (0,1]^n$ and $F = (0,1]^{n-1} \times (0,2]$. Notice that

$$F = (0,1]^n \sqcup (0,1]^{n-1} \times (1,2]. \tag{1.8}$$

Notice that $(0,1]^{n-1} \times (1,2] = \beta(E)$, where $\beta \colon \mathbb{R}^n \to \mathbb{R}^n$ is given by $\beta(x_1, \cdots, x_n) = (x_1, \cdots, x_n + 1)$, which is a movement. Thus, we get that

$$\begin{aligned} F &= E \sqcup \beta(E), \\ m(F) &= m(E) + m(\beta(E)), \\ &= 2m(E), \\ &= 2. \end{aligned} \tag{1.9}$$

However, we have seen that $m(F) \leqslant m(E)$, which leads us to the conclusion that $2 < 1$, which is clearly wrong.

We may state another similar problem. In our current formulation, it holds that $m\left(\bigsqcup_{i=1}^{k} E_i = \sum_{i=1}^{k} m(E_i)\right)$, but it would be interesting for this condition to hold more generally. After all, it might seem reasonable that the volume of a set made of countably many disjoint sets is the sum of their individual volumes. We would then be interested in finding a function $m \colon \mathbb{P}(\mathbb{R}^n) \to \bar{\mathbb{R}}$ such that

i. $m(E) \geqslant 0, \forall\, E \in \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}$;

ii. $m\left(\bigsqcup_{i=1}^{+\infty} E_i\right) = m(E) + m(F), \forall\, (E_i)_{i\in\mathbb{N}} \in \mathbb{P}(\mathbb{R}^n)^{\mathbb{N}}$;

iii. for any movement $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$, $(m \circ \beta)(E) = m(E), \forall\, E \in \mathbb{P}(\mathbb{R}^n)$;

iv. $m\left((0,1]^n\right) = 1$.

This new problem might seem silly, since its easier version was already unsolvable in dimensions equal to or higher than 3. However, we must acknowledge that such a restriction could force the solutions (that do exist) to dimensions $n = 1, 2$ to be unique. Vitali proved in 1905 that is doesn't happen and, in fact, such a problem admits no solution in any dimension. It might be instructive to see this in dimension $n = 1$.

Lemma 2:
$\forall\, y \in \mathbb{R}$, *let* $\beta_y\colon \mathbb{R} \to \mathbb{R}$ *denote the function* $\beta_y(x) = x + y$. β_y *is a movement.* □

Proof:
Let $x, y, z \in \mathbb{R}$.

$$\begin{aligned} d(\beta_z(x), \beta_z(y)) &= |\beta_z(x) - \beta_z(y)|, \\ &= |x + z - y - z|, \\ &= |x - y|, \\ &= d(x, y). \end{aligned} \tag{1.10}$$

This concludes the proof. ■

Lemma 3:
Suppose $m\colon \mathbb{P}(\mathbb{R}) \to \bar{\mathbb{R}}$ *is a function such that*

i. $m(E) \geqslant 0, \forall\, E \in \mathbb{P}(\mathbb{R}) \to \mathbb{R}$;

ii. $m\left(\bigsqcup_{i=1}^{+\infty} E_i\right) = m(E) + m(F), \forall\, (E_i)_{i\in\mathbb{N}} \in \mathbb{P}(\mathbb{R})^{\mathbb{N}}$;

iii. *for any movement* $\beta\colon \mathbb{R} \to \mathbb{R}$, $(m \circ \beta)(E) = m(E), \forall\, E \in \mathbb{P}(\mathbb{R})$;

iv. $m\left((0,1]\right) = 1$.

Then $m\left((a,b]\right) = b - a, \forall\, a, b \in \mathbb{R}, a < b$. □

Proof:
Suppose $a = 0, b \in \mathbb{N}^*$. If $b = 1$, the result is trivial. Let us assume then that $b > 1$. If, $\forall\, y \in \mathbb{R}$, we define $\beta_y\colon \mathbb{R} \to \mathbb{R}$ through $\beta_y(x) = x + y$ (Lemma 2 guarantees this is a

movement), we can write

$$
\begin{aligned}
(0,b] &= \bigsqcup_{n=0}^{b-1}(n,n+1], \\
&= \bigsqcup_{n=0}^{b-1} \beta_n((0,1]), \\
m((0,b]) &= \sum_{n=0}^{b-1} m(\beta_n((0,1])), \\
&= \sum_{n=0}^{b-1} m((0,1]), \\
&= b \cdot m((0,1]), \\
&= b. \qquad (1.11)
\end{aligned}
$$

Since we know $m((0,b-a]) = b-a$, it follows that $m((a,b]) = b-a$ whenever $b-a \in \mathbb{N}$, for β_a is a movement. Due to this same reason, we might without any loss of generality always assume $a = 0$ and simply prove the result for $(0,b]$, for $b \in \mathbb{R}_+$.

If $b \in \mathbb{Q}_+$, then we know that there are integers (which we can take to be natural numbers, since $b > 0$) p and q, $q \neq 0$, such that $b = \frac{p}{q}$. Notice that $m((0,qb]) = p$ due to our previous result. However

$$
\begin{aligned}
(0,qb] &= \bigsqcup_{n=1}^{q}(b\cdot(n-1), b\cdot n], \\
&= \bigsqcup_{n=1}^{q} \beta_{(n-1)b}((0,b]), \\
m((0,qb]) &= \sum_{n=1}^{q} m\left(\beta_{(n-1)b}((0,b])\right), \\
p &= \sum_{n=1}^{q} m\left((0,b]\right), \\
&= qm\left((0,b]\right), \\
m\left((0,b]\right) &= \frac{p}{q}. \qquad (1.12)
\end{aligned}
$$

This proves the result whenever $b-a \in \mathbb{Q}_+$. We must now deal with the case in which $b-a \in \mathbb{R}_+ \backslash \mathbb{Q}$.

Suppose $b \in \mathbb{R}_+ \backslash \mathbb{Q}$. Let us define a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ such that $y_{n+1} > y_n \geqslant y_0 = 0, \forall\, n \in \mathbb{N}$ and $y_n \to b$. This is possible, since $\mathbb{R}$ is the closure of $\mathbb{Q}$ in the standard topology. Thus, we may write

$$
\begin{aligned}
(0,b] &= \bigsqcup_{n=0}^{+\infty} (y_n, y_{n+1}], \\
m((0,b]) &= m\left(\bigsqcup_{n=0}^{+\infty} (y_n, y_{n+1}]\right), \\
&= \sum_{n=0}^{+\infty} m\left((y_n, y_{n+1}]\right), \\
&= \sum_{n=0}^{+\infty} y_{n+1} - y_n, \\
&= \lim_{N\to+\infty} \sum_{n=0}^{N} y_{n+1} - y_n, \\
&= \lim_{N\to+\infty} y_N - y_0, \\
&= \lim_{n\to+\infty} y_n, \\
&= b. \qquad (1.13)
\end{aligned}
$$

This concludes the proof. ■

Theorem 4:

There are no functions $m\colon \mathbb{P}(\mathbb{R}) \to \bar{\mathbb{R}}$ *such that*

i. $m(E) \geqslant 0, \forall\, E \in \mathbb{P}(\mathbb{R}) \to \mathbb{R}$;

ii. $m\left(\bigsqcup_{i=1}^{+\infty} E_i\right) = m(E) + m(F), \forall\, (E_i)_{i\in\mathbb{N}} \in \mathbb{P}(\mathbb{R})^{\mathbb{N}}$;

iii. *for any movement* $\beta\colon \mathbb{R} \to \mathbb{R}$, $(m \circ \beta)(E) = m(E), \forall\, E \in \mathbb{P}(\mathbb{R})$;

iv. $m\left((0,1]\right) = 1$. □

Proof:

Let us begin with a generic closed interval $[a,b]$. Notice that $m([a,b]) = b - a$. Indeed, since $[a,b] \subseteq (a-\epsilon, b], \forall\, \epsilon > 0$, we get that $m([a,b]) \leqslant m((a-\epsilon,b]) = b - a + \epsilon, \forall\, \epsilon > 0$. Thus, $m([a,b]) = b - a$.

Let us introduce an equivalence relation $\sim$ in $[0,1]$. We shall say $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. This is an equivalence relation indeed:

i. $x \sim x$, for $x - x = 0 \in \mathbb{Q}$;

ii. if $x \sim y$, then $x - y \in \mathbb{Q}$. Since $y - x = -(x-y) \in \mathbb{Q}$, we see that $y \sim x$;

iii. if $x \sim y$ and $y \sim z$, then $x - y, y - z \in \mathbb{Q}$. Therefore, $x - z = (x-y) + (y-z) \in \mathbb{Q}$ and we see that $x \sim z$.

Since $\sim$ is an equivalence relation on $[0,1]$, its equivalence classes constitute a disjoint covering of $[0,1]$. The Axiom of Choice allows us to create a set V comprised of one, and only one, element from each equivalence class.

$\forall\, y \in \mathbb{R}$, let $\beta_y : \mathbb{R} \to \mathbb{R}$ denote the movement $\beta_y(x) = x + y$. It is indeed a movement due to Lemma 2. Given $r, s \in \mathbb{Q}$, I claim that $\beta_r(V) \cap \beta_s(V) = \varnothing$ whenever $r \neq s$.

Suppose $x \in \beta_r(V) \cap \beta_s(V)$. Since $x \in \beta_r(V)$, there is $y \in V$ such that $x = y + r$. Since $x \in \beta_s(V)$, there is $z \in V$ such that $x = z + s$. As a consequence, we see that $z = y + (r - s)$. If we write $q = r - s$ (notice that $q \in \mathbb{Q}$), we see that $z - y = q$. Therefore, $z \sim y$. However, by definition of V, there are no two distinct elements of V that are equivalent. Therefore, we conclude $z = y$ and thus $r = s$.

I now claim that, if we define $\mathbb{Q}_1 = \mathbb{Q} \cap [-1, 1]$,

$$[0,1] \subseteq \bigsqcup_{r \in \mathbb{Q}_1} \beta_r(V) \subseteq [-1,2]. \tag{1.14}$$

Let $x \in V$. By construction, $V \subseteq [0,1]$, and therefore $x \leqslant 1$. Thus, if $r \in \mathbb{Q}_1$, we know that $x + r \leqslant 1$, for $r \leqslant 1$. Similarly, we know that $x \geqslant 0$ and $r \geqslant -1$. Thus, $x + r \geqslant -1$. This proves $\beta_r(V) \subseteq [-1,2], \forall\, r \in \mathbb{Q}_1$. Since $\beta_r(V) \cap \beta_s(V) = \varnothing$ whenever $r \neq s$, it follows that

$$\bigsqcup_{r \in \mathbb{Q}_1} \beta_r(V) \subseteq [-1,2]. \tag{1.15}$$

Let now $x \in [0,1]$. Then x is in some equivalence class E of $\sim$. We know there is one, and only one, $y \in E \cap V$. Since $x, y \in [0,1]$, we know that $x - y \in [-1,1]$. Since $x, y \in E$, $x - y \in \mathbb{Q}$. Thus, there is some $r \in \mathbb{Q}_1$ such that $x = y + r$. Therefore, $x \in \beta_r(V)$. We conclude that

$$[0,1] \subseteq \bigsqcup_{r \in \mathbb{Q}_1} \beta_r(V). \tag{1.16}$$

We now may see that, since $\mathbb{Q}$ is countable,

$$\begin{gathered}
[0,1] \subseteq \bigsqcup_{r \in \mathbb{Q}_1} \beta_r(V) \subseteq [-1,2], \\
m([0,1]) \leqslant m\left(\bigsqcup_{r \in \mathbb{Q}_1} \beta_r(V)\right) \leqslant m([-1,2]), \\
1 \leqslant \sum_{r \in \mathbb{Q}_1} m\left(\beta_r(V)\right) \leqslant 3, \\
1 \leqslant \sum_{r \in \mathbb{Q}_1} m\left(V\right) \leqslant 3.
\end{gathered} \tag{1.17}$$

This inequality fails for any value of $m(V)$, finite or infinite. If $m(V) = 0$, then we get that $1 < 0$. If $m(V) > 0$, then we get that $+\infty < 3$. Thus, it is impossible for such a function m to exist. ■

The interest in proving Theorem 4 comes in the search for an answer to the question "What did we do wrong?". Some axiom (or axioms) assumed in the definition of m must have been too strong, to the point of leading us into failure.

The proof to Theorem 4 showed us where the problem arose: there was some set V (the so-called Vitali Set) which could not be measure, and assuming it could be measured led to a contradiction. Similarly, with the Banach-Tarski Theorem, there would be no paradox if at least some of the sets that decompose E were actually impossible to measure.

Perhaps we were too greedy when trying to measure every possible set of $\mathbb{P}(\mathbb{R}^n)$. We shall now abandon our original goal of measuring every possible set and, instead, simply try to measure many sets.

2 Set Structures

Since measuring every element of $\mathbb{P}(\mathbb{R}^n)$ failed, we must now pursue the families of sets which we could measure. Thus, instead of dealing directly with the functions that attribute volume to sets, let us first be humble and prepare the stage.

Definition 5 [Symmetric Difference]**:**

Let M be a set. Given $E, F \in \mathbb{P}(M)$, we define the *symmetric difference* of E and F, denoted $E \triangle F$, is defined as

$$E \triangle F \equiv (E \cup F) \backslash (E \cap F). \tag{2.1}$$

♠

Lemma 6:

Let M *be a set,* $E, F \in \mathbb{P}(M)$*. Then*

$$E \triangle F = (E \backslash F) \sqcup (F \backslash E). \tag{2.2}$$

□

Proof:

$$\begin{aligned} E \triangle F &= (E \cup F) \backslash (E \cap F), \\ &= (E \cup F) \cap (E \cap F)^c, \\ &= \left(E \cap (E \cap F)^c\right) \cup \left(F \cap (E \cap F)^c\right), \\ &= (E \cap F^c) \cup (F \cap E^c), \\ &= (E \backslash F) \cup (F \backslash E). \end{aligned} \tag{2.3}$$

Since $E \backslash F \subseteq E$ and $F \backslash E = F \cap E^c \subseteq E^c$, we see that $(E \backslash F) \cap (F \backslash E) = \varnothing$. ■

Proposition 7:

Let M *be a set. Given* $E, F, G \in \mathbb{P}(M)$*, it holds that*

i. $E \triangle F = F \triangle E$*;*

ii. $E \triangle (F \triangle G) = (E \triangle F) \triangle G$. □

Proof:

For commutativity, one just needs to see that

$$\begin{aligned} E \triangle F &= (E \cup F) \backslash (E \cap F), \\ &= (F \cup E) \backslash (F \cup E), \\ &= F \triangle E. \end{aligned} \tag{2.4}$$

Associativity demands a bit more calculation:

$$
\begin{aligned}
(E\triangle F)\triangle G &= [(E\triangle F)\cap G^c]\cup\left[(E\triangle F)^c\cap G\right],\\
&= \{[(E\cap F^c)\cup(F\cap E^c)]\cap G^c\}\cup\left[(E\triangle F)^c\cap G\right],\\
&= [E\cap F^c\cap G^c]\cup[F\cap E^c\cap G^c]\cup\left\{[(E\cup F)\cap(E^c\cup F^c)]^c\cap G\right\},\\
&= [E\cap F^c\cap G^c]\cup[F\cap E^c\cap G^c]\cup\{[(E^c\cap F^c)\cup(E\cap F)]\cap G\},\\
&= [E\cap F^c\cap G^c]\cup[F\cap E^c\cap G^c]\cup[E^c\cap F^c\cap G]\cup[E\cap F\cap G],\\
&= [F^c\cap G^c\cap E]\cup[F\cap G^c\cap E^c]\cup[G\cap F^c\cap E^c]\cup[F\cap G\cap E],\\
&= [F\cap G^c\cap E^c]\cup[G\cap F^c\cap E^c]\cup[F^c\cap G^c\cap E]\cup[F\cap G\cap E],\\
&= [F\cap G^c\cap E^c]\cup[G\cap F^c\cap E^c]\cup\{[(F^c\cap G^c)\cup(F\cap G)]\cap E\},\\
&= [F\cap G^c\cap E^c]\cup[G\cap F^c\cap E^c]\cup\left\{[(F\cup G)\cap(F^c\cup G^c)]^c\cap E\right\},\\
&= \{[(F\cap G^c)\cup(G\cap F^c)]\cap E^c\}\cup\left[(F\triangle G)^c\cap E\right],\\
&= [(F\triangle G)\cap E^c]\cup\left[(F\triangle G)^c\cap E\right],\\
&= (F\triangle G)\triangle E,\\
&= E\triangle(F\triangle G).
\end{aligned}
\tag{2.5}
$$

This concludes the proof. ■

Lemma 8:
Let M *be a set and let* $E, F \in \mathbb{P}(M)$. *Suppose* $E \cap F = \varnothing$. *Then* $E\triangle F = E \cup F$. □

Proof:

$$
\begin{aligned}
E\triangle F &= (E\cup F)\backslash(E\cap F),\\
&= (E\cup F)\backslash\varnothing,\\
&= E\cup F.
\end{aligned}
\tag{2.6}
$$

This concludes the proof. ■

Lemma 9:
Let M *be a set and let* $E, F \in \mathbb{P}(M)$. *Suppose* $F \subseteq E$. *Then* $E\triangle F = E\backslash F$. □

Proof:

$$
\begin{aligned}
E\triangle F &= (E\cup F)\backslash(E\cap F),\\
&= E\backslash F.
\end{aligned}
\tag{2.7}
$$

This concludes the proof. ■

Since $\triangle$ is associative, we may define its action on a non-empty finite family of sets.

Definition 10 [Symmetric Difference of a Family]**:**
Let M be a set, $m \in \mathbb{N}$, $\mathcal{E} = \{E_i\}_{i=1}^m \subseteq \mathbb{P}(M)$. We define the *symmetric difference of the family* $\mathcal{E}$, $\triangle\mathcal{E}$, through

$$
\triangle\mathcal{E} = E_1\triangle\cdots\triangle E_m. \tag{2.8}
$$

We might also use the notation $\triangle_{n=1}^{m} E_n$. ♠

Proposition 11:

Let M *be a set,* $m \in \mathbb{N}$, $\{E_n\}_{n=1}^{m} \subseteq \mathbb{P}(M)$. *Then*

$$\mathop{\triangle}_{n=1}^{m} E_n = \{p \in M; p \in E_i \textit{ for an odd quantity of } E_i\}. \tag{2.9}$$

□

Proof:

Let us prove this by induction on m. We begin with $m = 2$.

We know that $E \triangle F = (E \cup F) \backslash (E \cap F)$. Thus, if x is in neither E nor F, then $x \notin E \triangle F$. If $x \in E \cap F$, then $x \notin E \triangle F$. If either $x \in E$ or $x \in F$, then $x \in E \cup F$, but $x \notin E \cap F$. Thus, $x \in E \triangle F$.

Suppose now $x \in \triangle_{n=1}^{m-1} E_n$. If $x \in E_m$, then $x \notin E_m \triangle \left[\triangle_{n=1}^{m-1} E_n\right] = \triangle_{n=1}^{m} E_n$ and x is in an even number of E_i (for it was in an odd number of E_i when E_m was ignored). A similar reasoning covers the three other cases and concludes the proof. ■

Notation:

Let M and N be sets. Let $f\colon M \to N$ be a function. Given $E \in \mathbb{P}(N)$, we denote its preimage by $f^{-1}(N)$. Given a family $\mathcal{E} \subseteq \mathbb{P}(N)$, we denote the family of the preimages of the elements of $\mathcal{E}$ through

$$f^{-1}(\mathcal{E}) = \left\{f^{-1}(E) \in \mathbb{P}(M); E \in \mathcal{E}\right\}. \tag{2.10}$$

♦

Definition 12 [Infima and Suprema of Sequences of Subsets]**:**

Let M be a set. Let $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$. We define the *infimum* $\inf_{n\in\mathbb{N}} E_n$ of $(E_n)_{n\in\mathbb{N}}$ through

$$\inf_{n\in\mathbb{N}} E_n = \bigcap_{n=0}^{+\infty} E_n. \tag{2.11}$$

Similarly, we define the *supremum* $\inf_{n\in\mathbb{N}} E_n$ of E_n through

$$\sup_{n\in\mathbb{N}} E_n = \bigcup_{n=0}^{+\infty} E_n. \tag{2.12}$$

♠

Remark:

Notice that $\inf_{n\in\mathbb{N}} E_n$ is the largest set contained in every E_n, whilst $\sup_{n\in\mathbb{N}} E_n$ is the smallest set containing every E_n. “Largest” and “smallest” should be understood in terms of the inclusion order. ♣

Definition 13 [Limits of Sequences of Subsets]**:**

Let M be a set. Let $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$. We define the *superior limit of* $(E_n)_{n\in\mathbb{N}}$, $\limsup_{n\to+\infty} E_n$, through

$$\limsup_{n\to+\infty} E_n = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} E_k. \tag{2.13}$$

Similarly, we define the *inferior limit of* $(E_n)_{n\in\mathbb{N}}$, $\liminf_{n\to+\infty} E_n$, through

$$\liminf_{n\to+\infty} E_n = \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} E_k. \tag{2.14}$$

If it holds that $\limsup_{n\to+\infty} E_n = \liminf_{n\to+\infty} E_n$, the sequence $(E_n)_{n\in\mathbb{N}}$ is said to *converge to* E, where $E = \limsup_{n\to+\infty} E_n = \liminf_{n\to+\infty} E_n$, and we write

$$\lim_{n\to+\infty} E_n = E. \tag{2.15}$$

♠

Proposition 14:

Let M *be a set. Let* $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$. *Then the following statements hold:*

i. $\liminf_{n\to+\infty} E_n = \{p \in M; \exists\, n \in \mathbb{N}; p \in E_k, \forall\, k \geqslant n\}$;

ii. $\limsup_{n\to+\infty} E_n = \{p \in M; \forall\, n \in \mathbb{N}, \exists\, k \geqslant n; p \in E_k\}$;

iii. $\liminf_{n\to+\infty} E_n \subseteq \limsup_{n\to+\infty} E_n$;

iv. $[\liminf_{n\to+\infty} E_n]^c = \limsup_{n\to+\infty} {E_n}^c$ *and* $\left[\limsup_{n\to+\infty} E_n\right]^c = \liminf_{n\to+\infty} {E_n}^c$;

v. $\left(\limsup_{n\to+\infty} E_n\right) \setminus (\liminf_{n\to+\infty} E_n) = \limsup_{n\to+\infty} (E_n \triangle E_{n+1})$. □

Proof:

i. Assume $p \in \liminf_{n\to+\infty} E_n$. Due to the definition of lim inf, this means that $p \in \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} E_k$. Thus, there is some $n \in \mathbb{N}$ such that $p \in \inf_{k\geqslant n} E_k$, *id est*, $p \in E_k, \forall\, k \geqslant n$.

   Suppose now that $\exists\, n \in \mathbb{N}; p \in E_k, \forall\, k \geqslant n$. Then it means $p \in \inf_{k\geqslant n} E_k$. As a consequence, $p \in \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} E_k = \liminf_{n\to+\infty} E_n$.

ii. Assume $p \in \limsup_{n\to+\infty} E_n$. Due to the definition of lim sup, this means that $p \in \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} E_k$. Thus, $\forall\, n \in \mathbb{N}$, $p \in \sup_{k\geqslant n} E_k$, *id est*, $\forall\, n \in \mathbb{N}, \exists\, k \geqslant n; p \in E_k$.

   Suppose now that $\forall\, n \in \mathbb{N}, \exists\, k \geqslant n; p \in E_k$. Then it means that $\forall\, n \in \mathbb{N}$, $p \in \sup_{k\geqslant n} E_k$. As a consequence, $p \in \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} E_k = \limsup_{n\to+\infty} E_n$.

iii. Assume that $\exists\, n \in \mathbb{N}; p \in E_k, \forall\, k \geqslant n$. Let $m \in \mathbb{N}$. If $m \leqslant n$, then we know there is $k \geqslant m$ such that $p \in E_k$, for every $k \geqslant n \geqslant m$ satisfies this. If $m > n$, then we know that every $k \geqslant m$ is such that $p \in E_k$, for every $k \geqslant m > n$ satisfies this. From the previous items follows the thesis.

iv. This is a consequence of de Morgan's laws:

$$\begin{aligned}
\left[\liminf_{n\to+\infty} E_n\right]^c &= \left[\bigcup_{n=0}^{\infty}\bigcap_{k=n}^{+\infty} E_k\right]^c, \\
&= \bigcap_{n=0}^{\infty}\left[\bigcap_{k=n}^{+\infty} E_k\right]^c, \\
&= \bigcap_{n=0}^{\infty}\bigcup_{k=n}^{+\infty} {E_k}^c, \\
&= \inf_{n\in\mathbb{N}}\sup_{k\geqslant n} {E_k}^c, \\
&= \limsup_{n\to+\infty} {E_k}^c;
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\left[\limsup_{n\to+\infty} E_n\right]^c &= \left[\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{+\infty} E_k\right]^c, \\
&= \bigcup_{n=0}^{\infty}\left[\bigcup_{k=n}^{+\infty} E_k\right]^c, \\
&= \bigcup_{n=0}^{\infty}\bigcap_{k=n}^{+\infty} {E_k}^c, \\
&= \sup_{n\in\mathbb{N}}\inf_{k\geqslant n} {E_k}^c, \\
&= \liminf_{n\to+\infty} {E_k}^c.
\end{aligned} \tag{2.17}$$

v. Let us assume $p \in \left[\limsup_{n\to+\infty} E_n\right] \setminus \left[\liminf_{n\to+\infty} E_n\right]$. Notice that, due to previous items,

$$\begin{aligned}
p \in \left[\limsup_{n\to+\infty} E_n\right] \setminus \left[\liminf_{n\to+\infty} E_n\right] &= \left[\limsup_{n\to+\infty} E_n\right] \cap \left[\liminf_{n\to+\infty} E_n\right]^c, \\
&= \left[\limsup_{n\to+\infty} E_n\right] \cap \left[\limsup_{n\to+\infty} {E_n}^c\right].
\end{aligned} \tag{2.18}$$

Thus, $\forall\, k \in \mathbb{N}, \exists\, n, m > k; p \in E_n \cap E_m^c$. We want to show that $\forall\, k \in \mathbb{N}, \exists\, n > k; p \in E_n \triangle E_{n+1}$.

Let us prove this by contradiction. Assume $\exists\, k \in \mathbb{N}; \forall\, n > k, p \notin E_n \triangle E_{n+1}$. If $p \notin E_n \triangle E_{n+1}$, then

$$\begin{aligned}
p \in [E_n \triangle E_{n+1}]^c &= \left[(E_n \cup E_{n+1}) \cap (E_n \cap E_{n+1})^c\right]^c, \\
&= (E_n \cup E_{n+1})^c \cup (E_n \cap E_{n+1}), \\
&= ({E_n}^c \cap {E_{n+1}}^c) \cup (E_n \cap E_{n+1}).
\end{aligned} \tag{2.19}$$

Suppose, for a given $n > p$, that $p \in E_n$. Since $p \notin E_n \triangle E_{n+1}$, it follows that $p \in ({E_n}^c \cap {E_{n+1}}^c) \cup (E_n \cap E_{n+1})$. The fact that $p \in E_n$ implies then that $p \in E_{n+1}$.

As a consequence, we see that $p \in E_m, \forall\, m \geqslant n$, and thus it can't hold that $\forall\, n \in \mathbb{N}, \exists\, m > n; p \in E_m^c$. Thus, $p \in \limsup_{n\to+\infty}(E_n \triangle E_{n+1})$, proving that

$$\left(\limsup_{n\to+\infty} E_n\right) \setminus \left(\liminf_{n\to+\infty} E_n\right) \subseteq \limsup_{n\to+\infty}\left(E_n \triangle E_{n+1}\right). \tag{2.20}$$

Let us suppose now that $p \in \limsup_{n\to+\infty}(E_n \triangle E_{n+1})$. Then $\forall\, k \in \mathbb{N}, \exists\, n > k; p \in E_n \triangle E_{n+1}$. Since

$$\left[\limsup_{n\to+\infty} E_n\right] \setminus \left[\liminf_{n\to+\infty} E_n\right] = \left[\limsup_{n\to+\infty} E_n\right] \cap \left[\limsup_{n\to+\infty} {E_n}^c\right], \tag{2.21}$$

we want to prove that $\forall\, k \in \mathbb{N}, \exists\, n, m > k; p \in E_n \cap E_m^c$.

Let us prove the contrapositive of this implication. Assuming $\exists\, k \in \mathbb{N}; \forall\, n, m > k, p \notin E_n \cap E_m^c$, *id est*, $p \in (E_n \cap E_m^c) = E_n^c \cup E_m$, we want to prove that $\exists\, k \in \mathbb{N}; \forall\, m > k, p \in [E_m \triangle E_{m+1}]^c$. In particular, we might take $m = n + 1$ and see that $\exists\, k \in \mathbb{N}; \forall\, n > k, p \in E_n^c \cup E_{n+1}$. We may also take $n = m + 1$ and see that $\exists\, k \in \mathbb{N}; \forall\, m > k, p \in E_{m+1}^c \cup E_m$. Thus, we see that $\exists\, k \in \mathbb{N}; \forall\, m > k$

$$\begin{aligned} p \in [E_m^c \cup E_{m+1}] \cap \left[E_{m+1}^c \cup E_m\right] &= [E_m \setminus E_{m+1}]^c \cap [E_{m+1} \setminus E_m]^c, \\ &= [(E_m \setminus E_{m+1}) \cup (E_{m+1} \setminus E_m]^c, \\ &= [E_m \triangle E_{m+1}]^c, \end{aligned} \tag{2.22}$$

and therefore we have proven the contrapositive of the statement that

$$\limsup_{n\to+\infty}\left(E_n \triangle E_{n+1}\right) \subseteq \left(\limsup_{n\to+\infty} E_n\right) \setminus \left(\liminf_{n\to+\infty} E_n\right). \tag{2.23}$$

Joining these two results, we conclude, as desired, that

$$\left(\limsup_{n\to+\infty} E_n\right) \setminus \left(\liminf_{n\to+\infty} E_n\right) = \limsup_{n\to+\infty}\left(E_n \triangle E_{n+1}\right), \tag{2.24}$$

concluding the proof. ■

Definition 15 [Monotone Sequences]**:**

Let M be a set and let $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$. The sequence $(E_n)_{n\in\mathbb{N}}$ is said to be *increasing* if, and only if, $E_n \subseteq E_{n+1}, \forall\, n \in \mathbb{N}$. It is said to be *decreasing* if, and only if, $E_{n+1} \subseteq E_n$. It is said to be *monotone* if, and only if, it is increasing or decreasing. ♠

Proposition 16:

Let M be a set and let $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$ be a monotone sequence. $(E_n)_{n\in\mathbb{N}}$ converges. If $(E_n)_{n\in\mathbb{N}}$ is increasing, then

$$\lim_{n\to+\infty} E_n = \sup_{n\in\mathbb{N}} E_n. \tag{2.25}$$

On the other hand, if $(E_n)_{n\in\mathbb{N}}$ is decreasing, then

$$\lim_{n\to+\infty} E_n = \inf_{n\in\mathbb{N}} E_n. \tag{2.26}$$

□

Proof:

We shall do the proof for the case in which $(E_n)_{n\in\mathbb{N}}$. The case for a decreasing sequence follows from a similar argument.

Notice that, since $E_n \subseteq E_{n+1}, \forall\, n \in \mathbb{N}$, it holds that $\inf_{k\geqslant n} E_k = E_n$. Indeed, $E_n \subseteq E_k, \forall\, k \geqslant n$, meaning $E_n \subseteq \inf_{k\geqslant n} E_k$. By the definition of infimum, $\inf_{k\geqslant n} E_k \subseteq E_n$, and the equality follows from these two inequalities.

Furthermore, given $m, n \in \mathbb{N}$, then we have that $\sup_{k\geqslant m} E_k = \sup_{k\geqslant n} E_k$. Indeed, let us assume without any loss of generality that $m \geqslant n$. Notice that $E_l \subseteq E_m \subseteq \sup_{k\geqslant m} E_k, \forall\, l \in \{l\}_{l=n}^{m}$, and thus

$$\begin{aligned} \sup_{k\geqslant n} E_k &= \bigcup_{l=n}^{m-1} E_l \cup \sup_{k\geqslant m} E_k, \\ &= \sup_{k\geqslant m} E_k. \end{aligned} \tag{2.27}$$

Finally, notice that

$$\begin{aligned} \limsup_{n\to+\infty} E_n &= \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} E_k, \\ &= \inf_{n\in\mathbb{N}} \sup_{k\geqslant 0} E_k, \\ &= \sup_{k\geqslant 0} E_k, \\ &= \sup_{n\in\mathbb{N}} E_n. \end{aligned} \tag{2.28}$$

On the other hand,

$$\begin{aligned} \liminf_{n\to+\infty} E_n &= \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} E_k, \\ &= \sup_{n\in\mathbb{N}} E_n, \\ &= \limsup_{n\to+\infty} E_n. \end{aligned} \tag{2.29}$$

Therefore, we may conclude that $(E_n)_{n\in\mathbb{N}}$ converges. Since

$$\limsup_{n\to+\infty} E_n = \liminf_{n\to+\infty} E_n = \sup_{n\in\mathbb{N}} E_n, \tag{2.30}$$

we see that

$$\lim_{n\to+\infty} E_n = \sup_{n\in\mathbb{N}} E_n, \tag{2.31}$$

concluding the proof. ■

Computing the limits of series of sets might be troublesome, but there is a way of converting such limits into numerical limits by employing characteristic functions.

Definition 17 [Characteristic Function of a Set]**:**

Let M be a set and $E \in \mathbb{P}(E)$. The *characteristic function of* E is the function $\chi_E \colon M \to \{0, 1\}$ defined through

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \tag{2.32}$$

♠

Proposition 18:

Let M be a set and $(E_n)_{n\in\mathbb{N}} \in \mathbb{P}(M)^{\mathbb{N}}$. For every $p \in M$ it holds that

$$\chi_{\limsup\limits_{n\to+\infty} E_n}(p) = \limsup_{n\to+\infty} \chi_{E_n}(p), \quad \chi_{\liminf\limits_{n\to+\infty} E_n}(p) = \liminf_{n\to+\infty} \chi_{E_n}(p). \tag{2.33}$$

□

Proof:

Suppose $\chi_{\limsup\limits_{n\to+\infty} E_n}(p) = 1$, *id est*, $p \in \limsup\limits_{n\to+\infty} E_n$. Then $p \in E_n$ for infinitely many $n \in \mathbb{N}$, *id est*, $\forall\, n \in \mathbb{N}, \exists\, k \geqslant n; p \in E_k$. Thus, $\forall\, n \in \mathbb{N}, \exists\, k \geqslant n; \chi_{E_k}(p) = 1$. It follows that

$$\begin{aligned} \limsup_{n\to+\infty} \chi_{E_n}(p) &= \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} \chi_{E_k}(p), \\ &= \inf_{n\in\mathbb{N}} 1, \\ &= 1, \\ &= \chi_{\limsup\limits_{n\to+\infty} E_n}(p). \end{aligned} \tag{2.34}$$

Suppose now $\chi_{\limsup\limits_{n\to+\infty} E_n}(p) = 0$, *id est*, $p \notin \limsup\limits_{n\to+\infty} E_n$. Then $\exists\, n_p \in \mathbb{N}; \forall\, k \geqslant n_p, p \notin E_k$. It follows that

$$\begin{aligned} \limsup_{n\to+\infty} \chi_{E_n}(p) &= \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} \chi_{E_k}(p), \\ &= \inf_{n\geqslant n_p} \sup_{k\geqslant n} \chi_{E_k}(p), \\ &= \inf_{n\geqslant n_p} 0, \\ &= 0, \\ &= \chi_{\limsup\limits_{n\to+\infty} E_n}(p). \end{aligned} \tag{2.35}$$

The proof for the inferior limit is analogous. ■

We might now define some algebraic structures in order to shape the collections of sets we want to work with.

Definition 19 [Group]**:**

Let G be a set and let $\cdot \colon G \times G \to G$ be a function. The pair $(G, \cdot)$ is said to be a *group* if, and only if,

i. $\forall\, g_1, g_2, g_3 \in G, g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ (associativity);

ii. $\exists\, e \in G; \forall\, g \in G, g \cdot e = e \cdot g = g$ (existence of neutral element);

iii. $\forall\, g \in G, \exists\, g^{-1} \in G; g \cdot g^{-1} = g^{-1} \cdot g = e$ (existence of inverse).

A group $(G, \cdot)$ is said to be *Abelian* or *commutative* if, and only if, $\forall\, g_1, g_2 \in G, g_1 \cdot g_2 = g_2 \cdot g_1$ (commutativity). ♠

Lemma 20:
Let $(G, \cdot)$ be a group. The following properties hold:

i. $\exists!\, e \in G; \forall\, g \in G, g \cdot e = e \cdot g = g;$

ii. $\forall\, g \in G, \exists!\, g^{-1} \in G; g \cdot g^{-1} = g^{-1} \cdot g = e.$ □

Proof:

i. Suppose e and e^* are neutral elements of G. Then it follows that

$$e = e \cdot e^* = e^*. \tag{2.36}$$

ii. Given $g \in G$, suppose g^{-1} and g^* are inverses of g. Then

$$g^* = e \cdot g^* = (g^{-1} \cdot g) \cdot g^* = g^{-1} \cdot (g \cdot g^*) = g^{-1} \cdot e = g^{-1}. \tag{2.37}$$

This concludes the proof. ■

Proposition 21:
Let M be a set. $(\mathbb{P}(M), \triangle)$ is an Abelian group. □

Proof:
By construction, $\triangle\colon \mathbb{P}(M) \times \mathbb{P}(M) \to \mathbb{P}(M)$ is a function. Proposition 7 guarantees $\triangle$ is associative and commutative. Notice that, given any $E \in \mathbb{P}(M)$,

$$\begin{aligned} E \triangle \varnothing &= (E \cup \varnothing) \backslash (E \cap \varnothing), \\ &= E \backslash \varnothing, \\ &= E. \end{aligned} \tag{2.38}$$

Since $\triangle$ is commutative, $\varnothing \triangle E = E$ as well. Thus, $\varnothing$ is a neutral element. Furthermore, notice that, given $E \in \mathbb{P}(M)$,

$$\begin{aligned} E \triangle E &= (E \cup E) \backslash (E \cap E), \\ &= E \backslash E, \\ &= \varnothing. \end{aligned} \tag{2.39}$$

Therefore, a given set $E \in \mathbb{P}(M)$ has itself as is its own inverse. This concludes the proof. ■

Definition 22 [Ring]**:**
Let R be a set and let $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$ be functions. The triple $(R, +, \cdot)$ is said to be a *ring* if, and only if,

i. $(R, +)$ is an Abelian group;

ii. $\bullet$ is associative: $\forall\, a, b, c \in R, a \bullet (b \cdot c) = (a \bullet b) \bullet c$;

iii. $\forall\, a, b, c \in R, a \bullet (b + c) = a \bullet b + a \bullet c$ (left distributive law of $\bullet$ over $+$);

iv. $\forall\, a, b, c \in R, (a + b) \bullet c = a \bullet c + b \bullet c$ (right distributive law of $\bullet$ over $+$).

A ring $(R, +, \bullet)$ is said to be *unital* if $\exists\, 1 \in R; \forall\, a \in R, a \bullet 1 = 1 \bullet a = a$, in which case 1 is called the ring's *unity*. It is said to be *commutative* if $\forall\, a, b \in R, a \bullet b = b \bullet a$.

The operation $+$ is commonly referred to as *addition*, while $\bullet$ is commonly referred to as *multiplication*. ♠

Notation:

In a ring $(R, +, \bullet)$, we denote the inverse additive element of some element $a \in R$ by $-a$. ♦

Lemma 23:

Let $(R, +, \bullet)$ *be a unital ring.* $\exists!\, 1 \in R; \forall\, a \in R, 1 \bullet a = a \bullet 1 = a.$ □

Proof:

Suppose 1 and 1^* are unities of $(R, +, \bullet)$. Then

$$1 = 1 \bullet 1^* = 1^*, \tag{2.40}$$

as desired. ■

Proposition 24:

Let M *be a set.* $(\mathbb{P}(M), \triangle, \cap)$ *is a unital commutative ring.* □

Proof:

Proposition 21 guarantees $(\mathbb{P}(M), \triangle)$ is an Abelian group. It is known from elementary set theory that $\cap$ is commutative and associative. Given $E \in \mathbb{P}(M)$, notice that

$$E \cap M = M \cap E = E. \tag{2.41}$$

Hence, if $(\mathbb{P}(M), \triangle, \cap)$ is a ring, it is a unital commutative ring with unity M. We now must only prove the distributive laws. Notice, though, that since $\cap$ is commutative, we may prove only one of the commutative laws, for the other will be implied.

Let $E, F, G \in \mathbb{P}(M)$. Then

$$\begin{aligned}
E \cap (F \triangle G) &= E \cap [(F \cup G) \cap (F^c \cup G^c)], \\
&= E \cap (F \cup G) \cap (F^c \cup G^c), \\
&= E \cap [F \cup G] \cap [E^c \cup F^c \cup G^c], \\
&= [(E \cap F) \cup (E \cap G)] \cap [E^c \cup F^c \cup E^c \cup G^c], \\
&= [(E \cap F) \cup (E \cap G)] \cap [(E \cap F)^c \cup (E \cap G)^c], \\
&= [(E \cap F) \cup (E \cap G)] \cap [(E \cap F) \cap (E \cap G)]^c, \\
&= (E \cap F) \triangle (E \cap G).
\end{aligned} \tag{2.42}$$

Therefore, $\cap$ distributes over $\triangle$, concluding the proof. ■

Definition 25 [Subring]**:**

Let $(R, +, \cdot)$ be a ring. A *subring* of $(R, +, \cdot)$ is a ring $(S, \oplus, \odot)$ such that $S \subseteq R$, $\oplus \colon S \times S \to S$ is given by $(a, b) \mapsto a + b$ and $\odot \colon S \times S \to S$ is given by $(a, b) \mapsto a \cdot b$. ♠

Notation:

In general, if $(R, +, \cdot)$ is a ring, we shall denote addition on a subring S by $+$ and multiplication by $\cdot$, despite the fact that we are indeed considering the restrictions of $+$ and $\cdot$ to $S \times S$. ♦

Lemma 26:

Let $(R, +, \cdot)$ *be a commutative ring and let* $(S, +, \cdot)$ *be a subring.* $(S, +, \cdot)$ *is commutative.* □

Proof:

We know that $\forall\, a, b \in R, a \cdot b = b \cdot a$. In particular, this holds for all $a, b \in S$. This concludes the proof. ■

Proposition 27:

Let $(R, +, \cdot)$ *be a ring. Let* 0 *denote the neutral additive element. Let* $S \subseteq R$. *Suppose* $\forall\, a, b \in S, a + b \in S$ *and* $a \cdot b \in S$, $0 \in S$ *and* $\forall\, a, -a \in S$. *Then it holds that* $(S, +, \cdot)$ *is a ring.* □

Proof:

Since $(R, +, \cdot)$ is a ring, we know $+$ is commutative and associative for all elements of R. This implies commutativity and associativity for all elements of S as a particular case. Since $0 \in S$ and $\forall\, a \in S, -a \in S$, it holds that $(S, +)$ is an Abelian group, for 0 is the neutral additive element for all elements of R (in particular, for all elements of S) and $-a$ is the additive inverse of a.

Since $\cdot$ is associative for all elements of R, it holds in particular that it is associative for all elements of S.

Since the distributive laws holds for all elements of R, it holds in particular that they hold for all elements of S. ■

Remark:

Notice that a subring of a unital ring doesn't need to be unital! It is not guaranteed by the definition that the unity will be an element of the subring. ♣

Definition 28 [Boolean Ring]**:**

Let M be a set and let $\mathcal{R} \subseteq \mathbb{P}(M)$ be non-empty. $\mathcal{R}$ is said to be a *ring of sets*, also known as a *Boolean ring*, over M if, and only if, $(\mathcal{R}, \triangle, \cap)$ is a subring of $(\mathbb{P}(M), \triangle, cap)$. ♠

Remark:

We shall often refer to Boolean rings as simply rings. ♣

Lemma 29:

Let $\mathcal{R}$ *be a ring over some set* M. *Then* $\varnothing \in \mathcal{R}$. □

Proof:

By definition, $\mathcal{R}$ is non-empty, and thus there is some $E \in \mathcal{R}$. Since $\mathcal{R}$ is a Boolean ring, we know $E \triangle E = \varnothing \in \mathcal{R}$. This concludes the proof. ■

Lemma 30:

Let M *be a set. Let* $E, F \in \mathbb{P}(M)$. *Then it holds that*

$$E \cup F = [E \triangle F] \triangle [E \cap F], \quad E \backslash F = E \triangle [E \cap F], \quad E \backslash F = [E \cup F] \triangle [E \triangle F]. \tag{2.43}$$

□

Proof:

Notice that $[E \triangle F] \cap [E \cap F] = \varnothing$:

$$\begin{aligned} [E \triangle F] \cap [E \cap F] &= [E \cup F] \cap [E \cap F]^c \cap [E \cap F], \\ &= \varnothing. \end{aligned} \tag{2.44}$$

Lemma 8 implies then that

$$\begin{aligned} [E \triangle F] \triangle [E \cap F] &= [E \triangle F] \sqcup [E \cap F], \\ &= [(E \cup F) \cap (E \cap F)^c] \cup (E \cap F), \\ &= (E \cup F) \cup (E \cap F), \\ &= E \cup F. \end{aligned} \tag{2.45}$$

For the second identity, notice that $E \cap F \subseteq E$. Thus, Lemma 9 yields

$$\begin{aligned} E \triangle [E \cap F] &= E \backslash (E \cap F), \\ &= E \backslash F. \end{aligned} \tag{2.46}$$

Finally, notice, for the third identity, that $E \triangle F \subseteq E \cup F$. Lemma 9 then implies

$$\begin{aligned} [E \cup F] \triangle [E \triangle F] &= [E \cup F] \backslash [E \triangle F], \\ &= (E \cup F) \cap [(E \cup F) \cap (E \cap F)^c]^c, \\ &= (E \cup F) \cap [(E \cup F)^c \cup (E \cap F)], \\ &= (E \cup F) \cap (E \cap F), \\ &= (E \cap F), \end{aligned} \tag{2.47}$$

as desired. ■

Theorem 31:

Let M *be a set and let* $\mathcal{R} \subseteq \mathbb{P}(M)$ *be non-empty. Then the following are equivalent:*

i. $\mathcal{R}$ *is a ring over* M;

ii. $\forall\, E, F \in \mathcal{R}, E \triangle F \in \mathcal{R}, E \cap F \in \mathcal{R}$;

iii. $\forall\, E, F \in \mathcal{R}, E \triangle F \in \mathcal{R}, E \cup F \in \mathcal{R}$;

iv. $\forall\, E, F \in \mathcal{R}, E \backslash F \in \mathcal{R}, E \cup F \in \mathcal{R}$. □

Proof:

i. $\Leftrightarrow$ ii. If i. holds, the definition of rind over a set automatically implies ii. Suppose then that ii. holds. Since $\mathcal{R}$ is non-empty, $\exists\, E \in \mathcal{R}$. Notice that $\forall\, E \in \mathcal{R}, -E \in \mathcal{R}$, for E is its own additive inverse. Furthermore, $E \triangle E = \varnothing \in \mathcal{R}$. Thus, all of the conditions of Proposition 27 are met, allowing us to conclude that $(\mathcal{R}, \triangle, \cap)$ is a subring of $(\mathbb{P}(M), \triangle, \cap)$, *id est*, $\mathcal{R}$ is a ring of sets.

ii. $\Rightarrow$ iii. It is clear that $E \triangle F \in \mathcal{R}$, for it is taken as a hypothesis. Lemma 30 ensures $E \cup F \in \mathcal{R}$.

iii. $\Rightarrow$ iv. It is clear that $E \cup F \in \mathcal{R}$. Lemma 30 ensures $E \backslash F \in \mathcal{R}$.

iv. $\Rightarrow$ ii. Lemma 6 guarantees $E \triangle F \in \mathcal{R}$. Since $E \cap F = [E \cup F] \backslash [E \triangle F]$, we see that $E \cap F \in \mathcal{R}$, concluding the proof. ■

Definition 32 [Boolean Algebra]**:**

Let M be a set and let $\mathcal{E} \subseteq \mathbb{P}(M)$ be a ring over M. $\mathcal{E}$ is said to be an *algebra of sets*, also known as a *Boolean algebra*, over M if, and only if, $\forall\, E \in \mathcal{E}, E^c \in \mathcal{E}$. ♠

Theorem 33:

Let M *be a set and let* $\mathcal{E} \subseteq \mathbb{P}(M)$ *be non-empty. Then the following are equivalent:*

i. $\mathcal{E}$ *is an algebra over* M*;*

ii. $\mathcal{E}$ *is a ring over* M *and* $M \in \mathcal{E}$*;*

iii. $\forall\, E, F \in \mathcal{E}, E^c \in \mathcal{E}, E \cap F \in \mathcal{E}$*;*

iv. $\forall\, E, F \in \mathcal{E}, E^c \in \mathcal{E}, E \cup F \in \mathcal{E}$. □

Proof:

i. $\Rightarrow$ ii. By the definition of algebra over a set, we know $\mathcal{E}$ is a ring. Lemma 29 guarantees $\varnothing \in \mathcal{E}$. Since $\mathcal{E}$ is an algebra, we see that $\varnothing^c = M \in \mathcal{E}$.

ii. $\Rightarrow$ iii. Theorem 31 guarantees that $\forall\, E, F \in \mathcal{E}, E \backslash F \in \mathcal{E}, E \cap F \in \mathcal{E}$. Let $E \in \mathcal{E}$. Since $M \in \mathcal{E}$, we see that $M \backslash E = E^c \in \mathcal{E}$.

iii. $\Rightarrow$ iv. $\forall\, E \in \mathcal{E}, E^c \in \mathcal{E}$, by hypothesis. Let $E, F \in \mathcal{E}$. Then $E^c, F^c \in \mathcal{E}$. Thus, $E^c \cap F^c \in \mathcal{E}$. Finally, $(E^c \cap F^c)^c = E \cup F \in \mathcal{E}$.

iv. $\Rightarrow$ i. Since $\mathcal{E}$ is non-empty, we know that $\forall\, E \in \mathcal{E}, -E \in \mathcal{E}$ (for E is its own additive inverse). We also know that $(E \cup E^c)^c = M^c = \varnothing \in \mathcal{E}$. Notice that $E \cap F = (E^c \cup F^c)^c \in \mathcal{E}$. Finally, $E \backslash F = E \cap F^c \in \mathcal{E}$. Proposition 27 and Theorem 31 guarantee then that $\mathcal{E}$ is a ring, and we know $E^c \in \mathcal{E}, \forall\, E \in \mathcal{E}$ by hypothesis. Hence, $\mathcal{E}$ is an algebra.

■

Algebras seem to be an interesting stage for us to develop Measure Theory. After all, if we know the volume of two different sets, we should also be able to know the volume of their union. We also expect to be able to know the volume of the whole space. Finally, if we know the volume of a set and some subset of it, we expect the volume of the different to be the difference of the volumes. Algebras (in fact, σ-algebras, which we shall define in an instant) shall eventually occur quite naturally within the theory.

We might also be interested in dealing with countable unions of sets, which motivates the following definition. Notice we are restricting ourselves to countable unions, for arbitrary unions would lead us to reobtaining $\mathbb{P}(M)$ far more easily, as we already know we can't measure every set.

Definition 34 [σ-sup-closed and σ-inf-closed]**:**

Let M be a set and let $\mathcal{E} \subseteq \mathbb{P}(M)$. $\mathcal{E}$ is said to be σ-*sup-closed* if, and only if, $\forall (E_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$, it holds that $\sup_{n\in\mathbb{N}} E_n \in \mathcal{E}$. Analogously, $\mathcal{E}$ is said to be σ-*inf-closed* if, and only if, $\forall (E_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$, it holds that $\inf_{n\in\mathbb{N}} E_n \in \mathcal{E}$.

Finally, $\mathcal{E}$ is said to be σ-*closed* if, and only if, it is simultaneously σ-sup-closed and σ-inf-closed. ♠

Definition 35 [σ-rings and σ-algebras]**:**

Let M be a set. A ring over M is said to be a σ-*ring* over M if, and only if, it is σ-sup-closed. Analogously, an algebra over M is said to be a σ-*algebra* over M if, and only if, it is σ-sup-closed. ♠

Proposition 36:

Let M be a set and let $\mathcal{R}$ be a ring over M. If $\mathcal{R}$ is σ-sup-closed, it is σ-inf-closed. Let $\mathcal{E}$ be an algebra over M. $\mathcal{E}$ is σ-sup-closed if, and only if, it is σ-inf-closed. □

Proof:

Let $(E_n)_{n\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$. We want to prove that $\inf_{n\in\mathbb{N}} E_n \in \mathcal{R}$.

Let us denote $E = \sup_{n\in\mathbb{N}} E_n$. Since $\mathcal{R}$ is σ-sup-closed, we know that $E \in \mathcal{R}$. Since $\mathcal{R}$ is a ring, we know that $F_n = E \setminus E_n \in \mathcal{R}, \forall\, n \in \mathbb{N}$. As a consequence, $\sup_{n\in\mathbb{N}} F_n \in \mathcal{R}$. Hence, $E\setminus[\sup_{n\in\mathbb{N}} F_n] \in \mathcal{R}$. However, notice that

$$\begin{aligned} E\setminus[\sup_{n\in\mathbb{N}} F_n] &= E \cap \left[\sup_{n\in\mathbb{N}}(E \cap E_n^c)\right]^c, \\ &= E \cap \left[\inf_{n\in\mathbb{N}}(E^c \cup E_n)\right], \\ &= E \cap \left[E^c \cup \inf_{n\in\mathbb{N}} E_n\right], \\ &= E \cap \inf_{n\in\mathbb{N}} E_n, \\ &= \inf_{n\in\mathbb{N}} E_n, \end{aligned} \tag{2.48}$$

where, in the last line, we used the fact that $\inf_{n\in\mathbb{N}} E_n \subseteq \sup_{n\in\mathbb{N}} E_n$. Thus, $\mathcal{R}$ is σ-inf-closed.

Let us now consider an algebra $\mathcal{E}$. Since $\mathcal{E}$ is closed under complements, we know that $E_n^c \in \mathcal{E}, \forall\, n \in \mathbb{N}$. We also know that $E = \sup_{n\in\mathbb{N}} E_n^c \in \mathcal{E}$. $E^c \in \mathcal{E}$. However,

$$E^c = \left[\sup_{n\in\mathbb{N}} E_n^c\right]^c = \inf_{n\in\mathbb{N}} E_n \in \mathcal{E}. \tag{2.49}$$

Hence, if $\mathcal{E}$ is σ-sup-closed, it is also σ-inf-closed. A similar proof applies to the reverse statement. ■

Corollary 37:

All σ-rings are σ-closed. □

Proof:

Straightforward from the definition of σ-ring and Proposition 36. ■

Corollary 38:

Let $\mathcal{R}$ be a σ-ring over a set M. *Let* $(E_n)_{n\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$. *It holds that*

$$\liminf_{n\to+\infty} E_n \in \mathcal{R}, \quad \limsup_{n\to+\infty} E_n \in \mathcal{R}. \tag{2.50}$$

□

Proof:

From Corollary 37 we know that $\forall\, n \in \mathbb{N}, F_n \equiv \inf_{k\geqslant n} E_k \in \mathcal{R}$. Corollary 37 also implies then that $\sup_{n\in\mathbb{N}} F_n \in \mathcal{R}$. Thus, we conclude that

$$\sup_{n\in\mathbb{N}} \inf_{k\geqslant n} E_k = \liminf_{n\to+\infty} E_n \in \mathcal{R}, \tag{2.51}$$

as desired. The proof for lim sup is analogous. ■

Theorem 39:

Let M *be a set and let* $\mathcal{E} \subseteq \mathbb{P}(M)$ *be a non-empty collection. The following are equivalent:*

i. $\mathcal{E}$ *is a σ-algebra;*

ii. $\forall\, E \in \mathcal{E}, E^c \in \mathcal{E}$ *and* $\mathcal{E}$ *is σ-sup-closed;*

iii. $\forall\, E \in \mathcal{E}, E^c \in \mathcal{E}$ *and* $\mathcal{E}$ *is σ-inf-closed.* □

Proof:

i. ⇒ ii. Holds by definition;

ii. ⇒ iii. Let $(E_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$. We know $E_n^c \in \mathcal{E}, \forall\, n \in \mathbb{N}$. Thus, $\sup_{n\in\mathbb{N}} E_n^c \in \mathcal{E}$. Finally, $\left[\sup_{n\in\mathbb{N}} E_n^c\right]^c = \inf_{n\in\mathbb{N}} E_n \in \mathcal{E}$.

iii. ⇒ i. Let $E, F \in \mathcal{E}$. σ-inf-closedness guarantees $E \cap F$, and we know $E^c \in \mathcal{E}$ by hypothesis. Theorem 33 guarantees that $\mathcal{E}$ is an algebra. Proposition 36 ensures $\mathcal{E}$ is a σ-algebra.

■

Given two sets M, N, a function $f\colon M \to N$ and a ring (algebra, σ-ring or σ-algebra) over N, we can "pull-it-back" to obtain a similar structure on M.

Proposition 40:

Let M *and* N *be sets and let* $f\colon M \to N$ *be a function. Suppose $\mathcal{R}$ is a ring, algebra, σ-ring or σ-algebra over* N. *Then*

$$f^{-1}(\mathcal{R}) \equiv \left\{f^{-1}(E); E \in \mathcal{R}\right\} \tag{2.52}$$

is a ring, algebra, σ-ring or σ-algebra over M. □

Proof:

Suppose $\mathcal{R}$ is a σ-ring. Let $F_1, F_2 \in f^{-1}(\mathcal{R})$. Then $\exists\, E_1, E_2 \in \mathcal{R}$ such that $F_1 = f^{-1}(E_1)$ and $F_2 = f^{-1}(E_2)$. We then have

$$\begin{aligned} F_1 \cup F_2 &= f^{-1}(E_1) \cup f^{-1}(E_2), \\ &= f^{-1}(E_1 \cup E_2) \in f^{-1}(\mathcal{R}). \end{aligned} \tag{2.53}$$

Similarly,

$$\begin{aligned} F_1 \backslash F_2 &= F_1 \cap F_2^c, \\ &= f^{-1}(E_1) \cap f^{-1}(E_2)^c, \\ &= f^{-1}(E_1 \cup E_2^c) \in f^{-1}(\mathcal{R}). \end{aligned} \tag{2.54}$$

Let $(F_n)_{n\in\mathbb{N}} \in f^{-1}(\mathcal{R})^{\mathbb{N}}$. Then there is $(E_n)_{n\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$ such that $F_n = f^{-1}(E_n), \forall\, n \in \mathbb{N}$. We then have

$$\begin{aligned} \sup_{n\in\mathbb{N}} F_n &= \sup_{n\in\mathbb{N}} f^{-1}(E_n), \\ &= f^{-1}\left(\sup_{n\in\mathbb{N}} E_n\right) \in f^{-1}(\mathcal{R}). \end{aligned} \tag{2.55}$$

Similar proofs apply for algebras. ■

Notation:

Let M and N be sets and let $f\colon M \to N$ be a function. Let $\mathcal{E} \subseteq \mathbb{P}(N)$ be an arbitrary collection. We denote the set of the preimages of elements of $\mathcal{E}$ by $f^{-1}(\mathcal{E})$, *id est*,

$$f^{-1}(\mathcal{E}) = \left\{f^{-1}(E); E \in \mathcal{E}\right\}. \tag{2.56}$$

◆

In a similar manner, given some structure defined on M, we are able to "push-it-forward" and obtain a similar structure on N.

Proposition 41:

Let M and N be sets and let $f\colon M \to N$ be a function. Suppose $\mathcal{R}$ is a ring, algebra, σ-ring or σ-algebra over M. Then

$$\mathcal{R}^* \equiv \left\{E \in \mathbb{P}(N); f^{-1}(E) \in \mathcal{R}\right\} \tag{2.57}$$

is the largest ring, algebra, σ-ring or σ-algebra over M such that $f^{-1}(\mathcal{R}^*) \subseteq \mathcal{R}$. □

Proof:

Let $E, F \in \mathcal{R}^*$. Then $f^{-1}(E), f^{-1}(F) \in \mathcal{R}$. Thus, $f^{-1}(E) \cup f^{-1}(F) = f^{-1}(E \cup F) \in \mathcal{R}$. Hence, $E \cup F \in \mathcal{R}^*$. Similar arguments can be made to prove $\mathcal{R}^*$ is a ring, algebra, σ-ring or σ-algebra over M.

Let us now prove that $\mathcal{R}^*$ is the largest such structure with the property that $f^{-1}(\mathcal{R}^*) \subseteq \mathcal{R}$. $f^{-1}(\mathcal{R}^*) = \mathcal{R}$ holds by the very definition of $\mathcal{R}^*$.

Suppose $\mathcal{E}$ is a ring, algebra, σ-ring or σ-algebra such that $f^{-1}(\mathcal{E}) \subseteq \mathcal{R}$. Let $E \in \mathcal{E}$. We know that $f^{-1}(E) \in \mathcal{R}$. Thus, $E \in \mathcal{R}^*$, by definition of $\mathcal{R}^*$. This concludes the proof. ■

We may specify a ring, algebra, σ-ring or σ-algebra by demanding it to be the smallest such structure containing a given set. This is possible due to the following result.

Lemma 42:

Let M be a set, Λ be an arbitrary collection of indices and let, $\forall\,\lambda \in \Lambda$, $\mathcal{R}_\lambda \subseteq \mathbb{P}(M)$ be a ring, algebra, σ-ring or σ-algebra. Then it holds that $\mathcal{R} \equiv \inf_{\lambda\in\Lambda} \mathcal{R}_\lambda$ is also a ring, algebra, σ-ring or σ-algebra. □

Proof:

We shall prove the result for σ-algebras. The remaining cases are similar.

Suppose $E \in \mathcal{R}$. Then $E \in \mathcal{R}_\lambda, \forall\,\lambda \in \Lambda$. Since all these collections are σ-algebras, $E^c \in \mathcal{R}_\lambda, \forall\,\lambda \in \Lambda$. Hence, $E^c \in \mathcal{R}$.

Assume now $(E_n)_{n\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$. Then $(E_n)_{n\in\mathbb{N}} \in \mathcal{R}_\lambda^{\mathbb{N}}, \forall\,\lambda \in \Lambda$. Since all $\mathcal{R}_\lambda$ are σ-algebras, it holds that $\sup_{n\in\mathbb{N}} E_n \in \mathcal{R}_\lambda, \forall\,\lambda \in \Lambda$. Hence, $\sup_{n\in\mathbb{N}} E_n \in \mathcal{R}$.

Theorem 39 guarantees $\mathcal{R}$ is a σ-algebra. ■

Definition 43 [Rings, Algebras, σ-rings and σ-algebras Generated by a Family]**:**

Let M be a set and let $\mathcal{F} \subseteq \mathbb{P}(M)$ be an arbitrary family of sets. The *ring generated by* $\mathcal{F}$ (analogously for an algebra, σ-ring or σ-algebra) is the smallest ring $\mathcal{R}$ containing the family $\mathcal{F}$, *id est*, the ring $\mathcal{R}(\mathcal{F})$ defined by

$$\mathcal{R}(\mathcal{F}) \equiv \inf\{\mathcal{R}; \mathcal{F} \subseteq \mathcal{R}, \mathcal{R} \text{ is a ring over } M\}. \tag{2.58}$$

Given a family $\mathcal{F}$, we denote the ring generated by $\mathcal{F}$ by $\mathcal{R}(\mathcal{F})$, the σ-ring generated by $\mathcal{F}$ by $\tilde{\sigma}(\mathcal{F})$, and the σ-algebra generated by $\mathcal{F}$ by $\sigma(\mathcal{F})$. ♠

Proposition 44:

Let M, N be sets, $f\colon M \to N$ be a function and $\mathcal{F} \subseteq \mathbb{P}(N)$. It holds that

$$f^{-1}(\sigma(\mathcal{F})) = \sigma\left(f^{-1}(\mathcal{F})\right), \tag{2.59}$$

with similar results for rings, algebras and σ-rings. □

Proof:

We shall prove the result for σ-algebras. The remaining cases are similar. ■

Finish proof!

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