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A Not So Brief Introduction to Topology

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Abstract

These are some study notes I've been developing while working on my undergraduate research project. This is a project still being developed, and pretty far from being finished (assuming it will ever be). This document, in particular, is a single chapter of a larger "book", aimed at the study of Hyperbolic Equations. You can find the "complete" work at http://fma.if.usp.br/~nickolas/pdf/Hyperbolic_Equations.pdf.

I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments and etc through my e-mail (nickolas@fma.if.usp.br). If you wish to have a look at more works, please check my personal website http://fma.if.usp.br/~nickolas.

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Contents

1 Metric Spaces

Mathematicians have a unique view of how things work. As an example, a mathematician would never say he or she has a unique view of how things work before proving that his or her view indeed exists, and only them would proceed to prove that such a view is unique. As a second example, suppose you wanted to discuss the way you measure the distance between two points in $\mathbb{R}^2$. Perhaps a physicist would simply pick a ruler and measure it. Or draw a right triangle and find the length of the hypotenuse with some elementary geometry. A mathematician would first ask "What do you mean by distance?"*.

In Mathematics, everything should be defined with absolutely no possibility of misinterpretation, and the concept of distance is no exception. What are the essential things about distance that really make it a distance? What if I wanted to measure distances in a different way? What should I never, ever, change?

One could simply say that the distance between two points is the radius of the circle that has one of them as the center and the other as a point on the circumference. But is this always what we mean by distance? If you thing about it for a minute, you will realize that there are indeed other meanings.

The easiest example can be given by thinking about a passenger and a driver on a taxicab. They might argue about whether the distance between the departure and arrival points should be calculated by using the definition above or some other idea. If the path

*Perhaps a mathematician would disagree with me, but as I said, they have a unique view of how things work.

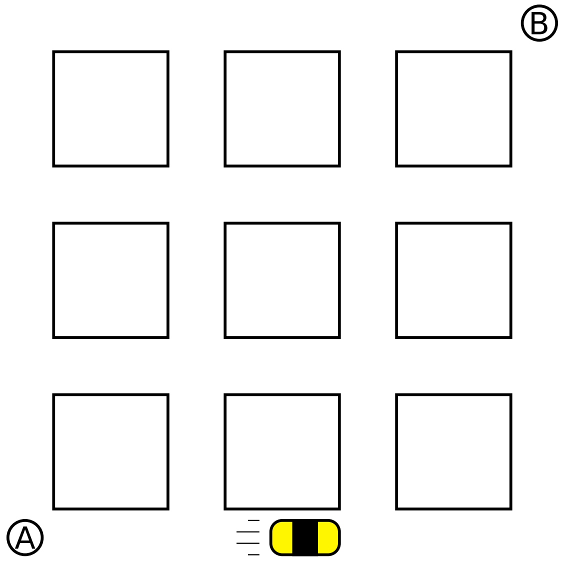

Figure 1. What is the distance between the departure point, A, and the arrival point, B? The passenger might wish to argue that it is something like 4.24 blocks away, but the taxicab driver certainly will prefer to say it is 6 blocks away.

was simply a straight line, there should be no argument whatsoever. However, if they had to pass through some blocks in order to arrive at their destination, as illustrated in Fig. 1, things could get complicated.

This example shows us that distance doesn't always means the same thing. However, not everything is lost! We might still search for the basic properties we expect a distance should have and *define* a distance to be such a thing.

One of the things we could ask is that if you are measuring the distance between two points on a certain space, this distance should never be negative. After all, would it even make sense to speak of a negative distance? Usually, the answer is no, and thus we are going to impose that no distance can be negative.

Another natural thing to require is that the distance between two points can only be zero if they are the same point (it would be weird if the distance between different points was zero, what would that even mean?). Besides, the distance between a point and itself should always be zero.

It is also natural for us to ask that the distance between point A and point B is the same as the distance between point B and point A. Perhaps transit officers would disagree with this idea, but we are going to cover this up in a minute. Mathematically, it makes sense for us to ask that distances are symmetrical.

Finally, the last requirement might seem a bit odd at a first glance, but give it a chance. We call it the triangle inequality. Pick two points A and B. For any third point C you choose, the distance between point A and point B can never be greater than the sum of the distances between point A and point C and between point C and point B. Intuitively, if you make a detour, the distance can never go down (it can stand still though).

Under these assumptions, we can finally give a proper definition to what a distance is. However, we usually don't call it a distance, but a metric.

Definition 1 [Metric Space]**:**

Let M be a non-empty set and $d : M \times M \to \mathbb{R}_+$ be a function such that, $\forall\, x, y, z \in M$:

i. $d(x, y) = 0 \Leftrightarrow x = y$;

ii. $d(x, y) = d(y, x)$ (symmetry);

iii. $d(x, y) \leqslant d(x, z) + d(y, z)$ (triangle inequality).

Under these conditions, we say that (M, d) is a *metric space* and d is said to be the *metric* defined on M. ♠

You might disagree with the hypotheses I made about what should be considered a distance. Perhaps you believe I should also require something else. Or you are a traffic officer and you believe I should not have asked that $d(x, y) = d(y, x)$*. And here comes an interesting part of Mathematics: you can simply use your own definition and work out its properties! Of course, I beg you not to call it a metric for the sake of clarity, but you can still find new results using your different hypotheses. I am going to use the definition I provided because it is the usual one and it shall yield the results I'm looking for, but I can't forbid you to work with something else.

Before we move on, it might be useful for us to give some examples of metric spaces.

Examples [$\mathbb{R}^n$]**:**

The Euclidean metric is the most usual metric in $\mathbb{R}^n$, and is defined by

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{1.1}$$

The taxicab metric we mentioned earlier without much rigour can be defined properly as

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|. \tag{1.2}$$

A third interesting metric can be defined as

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|. \tag{1.3}$$

The notation is not random, though it might seem: in fact,

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

*It is alright, you might not ask this one. If you use the triangle inequality as I stated ($d(x, y) \leqslant d(x, z) + d(y, z)$) and the condition that $d(x, y) = 0 \Leftrightarrow x = y$, you can prove that $d(x, y) = d(y, x)$ and that $d(x, y) \geqslant 0$. However, if you state the triangle inequality as $d(x, y) \leqslant d(x, z) + d(z, y)$, I can't guarantee that symmetry will still hold.

defines a metric on $\mathbb{R}^n$ for every $p \geqslant 1$, though we are not going to prove this fact. d_∞ is simply the limit with $p \to \infty$. For the special cases we've picked, most properties are pretty straightforward: we are taking sums of non-negative numbers (and a positive square root), these numbers can only sum to zero if $x_i = y_i$ for every i and the expression doesn't change if we exchange $\mathbf{x}$ and $\mathbf{y}$. The tricky property is the triangle inequality.

Let us begin by proving the triangle inequality in the case $n = 1$. You should notice that, under this assumption, $d_1(x, y) = d_2(x, y) = d_\infty(x, y) = |x - y|$.

We wish to prove that, given $x, y, z \in \mathbb{R}$, $|x - y| \leqslant |x - z| + |y - z|$. Without any loss of generality, we might assume $x \geqslant y$, and thus $|x - y| = x - y$. There are now three possibilities: $z \leqslant y \leqslant x$, $y \leqslant z \leqslant x$ and $y \leqslant x \leqslant z$.

If the first possibility holds, then

$$\begin{aligned} |x - z| + |y - z| &= x - z + y - z, \\ &= x - y + 2y - 2z, \\ &= |x - y| + 2|y - z|, \\ &\geqslant |x - y|. \end{aligned} \tag{1.4}$$

If the second possibility holds, we have

$$\begin{aligned} |x - z| + |y - z| &= x - z + z - y, \\ &= x - y, \\ &= |x - y|. \end{aligned} \tag{1.5}$$

For the third possibility,

$$\begin{aligned} |x - z| + |y - z| &= -x + z - y + z, \\ &= x - y + 2z - 2x, \\ &= |x - y| + 2|x - z|, \\ &\geqslant |x - y|. \end{aligned} \tag{1.6}$$

Thus, the triangle inequality holds for all the three metrics we gave when $n = 1$.

For arbitrary n, we might simply use this result for each i and realize that

$$\begin{aligned} |x_i - y_i| &\leqslant |x_i - z_i| + |y_i - z_i|, \\ \sum_{i=1}^{n} |x_i - y_i| &\leqslant \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |y_i - z_i|, \\ \therefore\ d_1(\mathbf{x}, \mathbf{y}) &\leqslant d_1(\mathbf{x}, \mathbf{z}) + d_1(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{1.7}$$

As for d_∞, let $1 \leqslant i, j, k \leqslant n$ be such that $|x_i - y_i| = \max_{1 \leqslant \alpha \leqslant n} |x_\alpha - y_\alpha|$, $\left|x_j - z_j\right| = \max_{1 \leqslant \beta \leqslant n} \left|x_\beta - z_\beta\right|$ and $|y_k - z_k| = \max_{1 \leqslant \gamma \leqslant n} |y_\gamma - z_\gamma|$. Notice that, by definition of j and k, it holds that $\left|x_j - z_j\right| \geqslant |x_i - z_i|$ and $|y_k - z_k| \geqslant |y_i - z_i|$ Using the result we have for $n = 1$, we may write

$$\begin{aligned}
|x_i - y_i| &\leqslant |x_i - z_i| + |y_i - z_i|, \\
&\leqslant \left|x_j - z_j\right| + |y_k - z_k|, \\
\max_{1\leqslant\alpha\leqslant n} |x_\alpha - y_\alpha| &\leqslant \max_{1\leqslant\beta\leqslant n} \left|x_\beta - z_\beta\right| + \max_{1\leqslant\gamma\leqslant n} |y_\gamma - z_\gamma|, \\
\therefore d_\infty(\mathbf{x}, \mathbf{y}) &\leqslant d_\infty(\mathbf{x}, \mathbf{z}) + d_\infty(\mathbf{y}, \mathbf{z}).
\end{aligned} \tag{1.8}$$

Finally, we need to prove the same inequality for d_2. You might recall from linear algebra that the usual norm on $\mathbb{R}^n$ looks pretty much like d_2*. If we define an inner product on $\mathbb{R}^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i, \tag{1.9}$$

then we know that the norm induced by this inner product is going to be

$$\|\mathbf{x}\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{1.10}$$

Notice then that $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

The triangle inequality can then be proved by making use of either the triangle inequality for the norm or Cauchy-Schwartz inequality (which also can be used to proved that inner products do induce a norm). We are going to use the latter.

The Cauchy-Schwartz inequality can be written (for a vector space over the real numbers) as

$$\langle \mathbf{x}, \mathbf{y} \rangle \leqslant \|\mathbf{x}\| \cdot \|\mathbf{y}\|. \tag{1.11}$$

We might write

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle, \\
\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle, \\
&= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2, \\
&\leqslant \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{y}\|^2, \\
&\leqslant \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.
\end{aligned} \tag{1.12}$$

Eq. (1.12) proves the triangle inequality for norms induced by an inner product. By using $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ and $(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})$, one can prove the triangle inequality for d_2. ♥

Examples [Function Spaces]**:**

We might as well give some examples concerning function spaces. Let $\mathcal{C}^0([0, 1])$ be the

*This is an example of the fact that, if $\|\cdot\|$ is a norm on a vector space V, then $d(u, v) := \|u - v\|$ is a metric on V. I invite you to prove this theorem.

set of all continuous functions $f\colon [0,1] \to [0,1]$. Then $(\mathcal{C}^0([0,1], d_p)$ with $p \geqslant 1$ or $p = \infty$ is a metric space, with d_p defined by

$$d_p(f,g) := \left[\int_0^1 |f(x) - g(x)|^p \, dx\right]^{\frac{1}{p}}, \tag{1.13}$$

for finite p or

$$d_\infty(f,g) := \lim_{p\to\infty} d_p(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|, \tag{1.14}$$

for infinite p.

Since the functions are continuous, the existence of a point x_0 such that $f(x_0) \neq g(x_0)$ implies the existence of an open interval $(a_0, b_0) \subseteq \mathbb{R}$ such that $x_0 \in (a_0, b_0)$ and $f(x) \neq g(x), \forall\, x \in (a_0, b_0)$.

Indeed, since f and g are continuous, so is $h(x) \equiv f(x) - g(x)$. Supposing, without any loss of generality, that $f(x_0) > g(x_0)$, we know that

$$\forall\, \epsilon > 0, \exists\, \delta > 0; |x - x_0| < \delta \Rightarrow |h(x) - h(x_0)| < \epsilon. \tag{1.15}$$

If we pick $\epsilon = \frac{1}{2}h(x_0)$, then there is $\delta > 0$ such that, if $x \in (x_0 - \delta, x_0 + \delta)$, then

$$-\frac{1}{2}h(x_0) < h(x) - h(x_0) < \frac{1}{2}h(x_0). \tag{1.16}$$

Therefore, we found an open interval such that

$$\frac{1}{2}h(x_0) < h(x) < \frac{3}{2}h(x_0) \tag{1.17}$$

for every $x \in (a_0, b_0)$, with $a_0 = x_0 - \delta$ and $b_0 = x_0 + \delta$.

Notice then that the integral of $|f(x) - g(x)|$ over this open interval is certainly positive (never zero), while the integral over the rest of the domain is certainly not negative (since we are integrating a non-negative function). Thus, $d_p(f,g) > 0$ if $f(x) \neq g(x)$. The case $p \to \infty$ is even easier: if there is a point in which $|f(x_0) - g(x_0)| \neq 0$, then $d_\infty(f,g) \geqslant |f(x_0) - g(x_0)|$, since 0 is the smallest value $|f(x) - g(x)|$ can assume.

The reverse implication and the symmetry property are left as exercises. The proof of the triangle inequality is omitted. ♥

The same process that allowed us to find a satisfying definition to what is a metric allows us to generalize the concept of an open set. Of course, you might be wondering: why would I ever care about open sets so much that I would want to make this concept more general?

As we shall see in a moment, open sets are extremely related to the notions of limits and continuity of a function. Indeed, we are able to define what is a continuous function without ever needing to speak about what is a limit, and this definition would still be consistent with the usual definition based on limits (actually, it would extend this notion!).

For us to speak of continuity, we will need for the domain and range to have some *topological* structure. This is just mathematical slang to say that they should obey some conditions in order for the terms "limit", "continuity" and everything else to make sense.

That being said, let's move on to work out some properties of open sets in metric spaces.

Perhaps you remember from advanced calculus courses what is an open ball in $\mathbb{R}^n$ and so on. In terms of metric spaces, these ideas can be expressed in a more general form.

Definition 2 [Open Ball]**:**

Let (M, d) be a metric space. We define the *open ball with center* $x \in M$ *and radius* $r \in \mathbb{R}_+$ as the set

$$\mathcal{B}_r(x) \equiv \mathcal{B}(x, r) := \{y \in M; d(x, y) < r\}. \quad ♠$$

As you should notice, this definition is essentially the same you will find in calculus courses, the only difference being it is written in terms of a general metric. In calculus, one would use usually the Euclidean metric.

The notions of interior point of a set and open sets are still the same.

Definition 3 [Interior Point]**:**

Let (M, d) be a metric space and let $X \subseteq M$ be a set. We say a point $x \in X$ is an *interior point* of X if, and only if, $\exists\, r > 0; \mathcal{B}_r(x) \subseteq X$. ♠

Definition 4 [Open Set]**:**

Let (M, d) be a metric space and let $X \subseteq M$ be a set. We say X is *open* if, and only if, every point $x \in X$ is an interior point of X. ♠

Lemma 5:

Given any metric space (M, d)*, every open ball is an open set.* □

Proof:

Let $x_0 \in M$ and $r \in \mathbb{R}_+$. We want to prove that $\mathcal{B}_r(x_0)$ is an open set.

Let $x \in \mathcal{B}_r(x_0)$ and let $s \equiv r - d(x_0, x)$. Since $x \in \mathcal{B}_r(x_0)$, $d(x, x_0) < r$ and thus $s > 0$. I claim $\mathcal{B}_s(x) \subseteq \mathcal{B}_r(x_0)$.

Let $x' \in \mathcal{B}_s(x)$. Due to the triangle inequality, we have that

$$\begin{aligned} d(x_0, x') &\leqslant d(x_0, x) + d(x, x'), \\ &< d(x_0, x) + s, \\ &= d(x_0, x) + r - d(x, x_0), \\ &= r. \end{aligned} \tag{1.18}$$

Since $d(x_0, x') < r$, we know that $x' \in \mathcal{B}_r(x_0)$. Therefore, $\mathcal{B}_s(x) \subseteq \mathcal{B}_r(x_0)$ and thus every point of $\mathcal{B}_r(x_0)$ is an interior point, *id est*, $\mathcal{B}_r(x_0)$ is an open set. ■

As said before, open sets are extremely interesting due to their relation to continuity. In order to show this, we must of course know what do we mean by continuity.

Definition 6 [Continuous Function]**:**

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. f is said to be *continuous at a point* x_0 if, and only if, $\forall\, \epsilon > 0, \exists\, \delta > 0; d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$. If f is continuous at every $x_0 \in X$, then we say f is a *continuous function*. ♠

Theorem 7:

Let (X, d_X) *and* (Y, d_Y) *be metric spaces and* $f\colon X \to Y$ *be a function. Then* f *is a continuous function if, and only if, the preimage* $f^{-1}(A)$ *of* A *is an open set for every open set* $A \subseteq Y$. □

Proof:

Let f be a continuous function and let $A \subseteq Y$ be an open set. Suppose $x_0 \in X$ is such that $f(x_0) \in A$. Since A is an open set, we know that there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon(f(x_0)) \subseteq A$.

However, since f is continuous, we know that there is $\delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$, *id est*, $x \in \mathcal{B}_\delta(x_0) \Rightarrow f(x) \in \mathcal{B}_\epsilon(f(x_0)) \subseteq A$. Thus, we know that the image of $\mathcal{B}_\delta(x_0)$ under f, $f(\mathcal{B}_\delta(x_0))$, is contained in A. Therefore, $\mathcal{B}_\delta(x_0) \subseteq f^{-1}(A)$, which means that x_0 is an interior point of $f^{-1}(A)$. Since the same argument applies to any point with image in A, it follows that $f^{-1}(A)$ is open for every open set $A \subseteq Y$, since every point of $f^{-1}(A)$ is an interior point.

If A happened to be empty, then $f^{-1}(A)$ would also be empty and, therefore, trivially open.

Suppose now that, for every open set $A \subseteq Y$, $f^{-1}(A)$ is an open set as well. We wish to prove that f is continuous.

Let $x_0 \in X$ and $\epsilon > 0$. Due to Lemma 5, we know that $\mathcal{B}_\epsilon(f(x_0))$ is an open set, and thus its preimage is an open set as well by hypothesis. Since $x_0 \in f^{-1}(\mathcal{B}_\epsilon(f(x_0)))$, it has to be an interior point and thus there is $\delta > 0$ such that $\mathcal{B}_\delta(x_0) \subseteq f^{-1}(\mathcal{B}_\epsilon(f(x_0)))$. Thus, $\forall\,\epsilon, \exists\,\delta$ such that

$$\begin{aligned} d_X(x, x_0) < \delta &\Rightarrow x \in \mathcal{B}_\delta(x_0), \\ &\Rightarrow x \in f^{-1}(\mathcal{B}_\epsilon(f(x_0))), \\ &\Rightarrow f(x) \in \mathcal{B}_\epsilon(f(x_0)), \\ &\Rightarrow d_Y(f(x), f(x_0)) < \epsilon. \end{aligned} \tag{1.19}$$

Therefore, f is continuous. ■

Theorem 7 shows that studying open sets allows us to understand continuous functions better. Furthermore, if we are able to generalize the concept of an open set, we will be able to extend the definition of continuity beyond metric spaces.

In order to do so, we can prove some more properties about the collection of all open sets in a given metric space (M, d). Such a collection is called a *topology* in M.

Theorem 8:

Let (M, d) *be a metric space,* $\tau \subseteq \mathbb{P}(M)$*, where* $\mathbb{P}(M)$ *denotes the powerset of* M*, and* Λ *be an arbitrary set of indexes. Then it holds that*

i. $\varnothing, M \in \tau$;

ii. $X, Y \in \tau \Rightarrow X \cap Y \in \tau$;

iii. $X_\lambda \in \tau, \forall\,\lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} X_\lambda \in \tau$. □

Proof:

i. since there are no elements in $\varnothing$, it is vacuously true that every point in $\varnothing$ is an interior point. Let $x_0 \in M$ and let $r > 0$. Since $\mathcal{B}_r(x_0)$ is defined as a subset of M, of course $\mathcal{B}_r(x_0) \subseteq M$, and thus x_0 is an interior point of M. Therefore, every point of M is an interior point, which means that M is an open set;

ii. let $x_0 \in X \cap Y$. Since $x_0 \in X$, there exists $r_1 > 0$ such that $\mathcal{B}_{r_1}(x_0) \subseteq X$. Similarly, since $x_0 \in Y$, there exists $r_2 > 0$ such that $\mathcal{B}_{r_2}(x_0) \subseteq Y$. Let $r = \min\{r_1, r_2\}$. Then $\mathcal{B}_r(x_0) \subseteq \mathcal{B}_{r_1}(x_0) \subseteq X$ and $\mathcal{B}_r(x_0) \subseteq \mathcal{B}_{r_2}(x_0) \subseteq Y$. Therefore, $\mathcal{B}_r(x_0) \subseteq X \cap Y$, and thus x_0 is an interior point of $X \cap Y$. Since the argument holds for any $x_0 \in X \cap Y$, the set must be open. Of course, if $X \cap Y = \varnothing$, then $X \cap Y$ is an open set due to the previous item;

iii. let $x_0 \in \bigcup_{\lambda \in \Lambda} X_\lambda$. Then $x_0 \in X_\lambda$ for some $\lambda \in \Lambda$. Since X_λ is open, there exists $r > 0$ such that $\mathcal{B}_r(x_0) \subseteq X_\lambda \subseteq \bigcup_{\lambda \in \Lambda} X_\lambda$. Therefore, x_0 is an interior point of $\bigcup_{\lambda \in \Lambda} X_\lambda$. Of course, if either $\Lambda = \varnothing$ or $X_\lambda = \varnothing, \forall \lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} X_\lambda = \varnothing$ and the set is open due to the first item. ■

This structure suggests a way of extending the definition of what is an open set.

2 Topological Spaces

Previously, we had to enumerate the basic properties we believed to be necessary to define what is distance. Now, in order to extend the definition of what is an open set, we might simply build upon the conclusions of Theorem 8.

Definition 9 [Topological Space]**:**
Let X be a non-empty set, let $\tau \subseteq \mathbb{P}(X)$ be a set and let Λ be an arbitrary set of indexes. τ is said to be a *topology* on X and (X, τ) is said to be a *topological space* if, and only if, the following axioms hold:

i. $\varnothing, X \in \tau$;

ii. $A, B \in \tau \Rightarrow A \cap B \in \tau$;

iii. $A_\lambda \in \tau, \forall \lambda \in \Lambda \Rightarrow \displaystyle\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$. ♠

Of course, now that we don't have any metric we need to update our definition of what is an open set. As our definition of topological space might suggest, the open sets are simply the elements of the topology we are considering.

Definition 10 [Open Set]**:**
Let (X, τ) be a topological space. We say a set $A \subseteq X$ is an *open set* if, and only if, $A \in \tau$. ♠

Of course, given a fixed set X, the open sets on (X, τ) might be very different depending on the topology we choose. As the most elementary example, we proved in the previous section that every metric induces a topology in a metric space (we call it a *metric topology*), and the same set might admit many different metrics.

Examples [Some Topological Spaces]**:**

The first example of a topological space one might pick is defined by "What happens if I let every set be an open set?". Given a non-empty set X, just set $\tau = \mathbb{P}(X)$. All axioms hold and you have at your hands the *discrete topology*. Curiously, this is also a metric topology, and is induced by the trivial metric:

$$d_T(x, y) = \begin{cases} 0, \text{ if } x = y, \\ 1, \text{ if } x \neq y. \end{cases} \tag{2.1}$$

The next thing that might come to your mind is "If I can make every set be an open set, can I choose a topology so that no set is open?". The answer is no, since one of the requirements for τ to be a topology is that $\varnothing, X \in \tau$. Nevertheless, $\tau = \{\varnothing, X\}$ does constitute a topology, and is called the *trivial topology* or the *indiscrete topology*.

Given a set X and a set $A \subseteq X$, $\tau \equiv \{B \subseteq X; A \subseteq B\} \cup \{\varnothing\}$ defines a topology in X. Indeed, $\varnothing, X \in \tau$ trivially. Given any two sets $B, C \in \tau$, $B \cap C \subseteq X$ and $A \subseteq B \cap C$ (for $A \subseteq B \subseteq X$ and $A \subseteq C \subseteq X$). Finally, given a set of indexes Λ, a collection of sets $B_\lambda \in \tau$ and any particular $\lambda_0 \in \Lambda$, we have that $A \subseteq B_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} B_\lambda \subseteq X$, the last relation being valid due to the fact that $B_\lambda \subseteq X$ for every $\lambda \in \Lambda$.

Let (X, τ) be a topological space and $A \subseteq X$. Then $\tau' \equiv \{A \cap O; O \in \tau\}$ is a topology on A, the so called *subspace* (also called *induced* or *relative*) *topology*. Indeed, since $\varnothing, X \in \tau$, $\varnothing = A \cap \varnothing \subseteq \tau'$ and $A = A \cap X \subseteq \tau'$. Given $B, C \in \tau'$, we know by the definition of τ' that there are $O_1, O_2 \in \tau$ such that $B = A \cap O_1$ and $C = A \cap O_2$, and thus $B \cap C = (A \cap O_1) \cap (A \cap O_2) = A \cap (O_1 \cap O_2)$. Since τ is a topology, $O_1 \cap O_2 \in \tau$ and it follows that $B \cap C \in \tau'$. Finally, given a set of arbitrary indexes Λ and a collection of sets $B_\lambda \in \tau', \forall \lambda \in \Lambda$, we know there is a collection $O_\lambda \in \tau; B_\lambda = A \cap O_\lambda, \forall \lambda \in \Lambda$. Therefore, $\bigcup_{\lambda \in \Lambda} B_\lambda = \bigcup_{\lambda \in \Lambda} A \cap O_\lambda = A \cap \bigcup_{\lambda \in \Lambda} O_\lambda$. Since τ is a topology, $\bigcup_{\lambda \in \Lambda} O_\lambda \in \tau$ and thus $\bigcup_{\lambda \in \Lambda} B_\lambda \in \tau'$, proving our claim. ♥

As you see, different topologies might have more or less open sets than others, and such comparisons lead us to the following definition:

Definition 11 [Finer, Coarser and Comparable]**:**

Let X be a set and let τ and τ' be topologies on X. If $\tau \subseteq \tau'$, we say that τ' is *finer* than τ. If $\tau \subset \tau'$, we say τ' is *strictly finer* than τ. Under the same assumptions, we say τ is *coarser*, or *strictly coarser*, than τ'. Whenever $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$ we say τ and τ' are *comparable*. ♠

In fact, it is even possible to define the finest topology containing a given set or the coarsest topology contained within a given set.

Theorem 12:

Let X be a set, Λ be an arbitrary set of indexes and $(\tau_\lambda)_{\lambda \in \Lambda}$ be a family of topologies on X. Then $\bigcap_{\lambda \in \Lambda} \tau_\lambda$ is a topology on X. □

Proof:

Since $\varnothing, X \in \tau_\lambda, \forall \lambda \in \Lambda$, we know that $\varnothing, X \in \bigcap_{\lambda \in \Lambda} \tau_\lambda$.

Suppose that $O_1, O_2 \in \bigcap_{\lambda \in \Lambda} \tau_\lambda$. Then we know that $O_1, O_2 \in \tau_\lambda, \forall \lambda \in \Lambda$. Since every τ_λ is a topology, we know that $O_1 \cap O_2 \in \tau_\lambda, \forall \lambda \in \Lambda$. Therefore, it follows that $O_1 \cap O_2 \in \bigcap_{\lambda \in \Lambda} \tau_\lambda$.

Let Ω be an arbitrary set of indexes and $O_\omega \in \bigcap_{\lambda\in\Lambda} \tau_\lambda, \forall\, \omega \in \Omega$. Then we know that $O_\omega \in \tau_\lambda, \forall\, \omega \in \Omega, \forall\, \lambda \in \Lambda$. Since every τ_λ is a topology, it follows that $\bigcup_{\omega\in\Omega} O_\omega \in \tau_\lambda, \forall\, \lambda \in \Lambda$. Therefore, we conclude that $\bigcup_{\omega\in\Omega} O_\omega \in \bigcap_{\lambda\in\Lambda} \tau_\lambda$. This concludes the proof. ■

Remark:

If the intersection of topologies is still a topology, a natural question one could ask is whether the union of topologies is a topology.

Let $X = \{a, b, c\}$ be a set. Consider the topologies

$$\tau_1 = \{\varnothing, X, \{a\}, \{a, b\}\}, \quad \tau_2 = \{\varnothing, X, \{a\}, \{b, c\}\}. \tag{2.2}$$

I will leave to you the pleasure of proving that τ_1 and τ_2 are indeed topologies. Notice that $\tau \equiv \tau_1 \cup \tau_2 = \{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}\}$ is *not* a topology. Indeed, topologies are always closed under finite intersections, and $\{a, b\} \cap \{b, c\} = \{b\} \notin \tau$, even though $\{a, b\}, \{b, c\} \in \tau$. Thus, τ is not a topology. ♣

Proposition 13:

Let X be a set, Λ be an arbitrary set of indexes and $\{\tau_\lambda\}_{\lambda\in\Lambda}$ be a family of topologies on X. Then it holds that:

i. *there exists the coarsest topology in the family of topologies finer than $\tau_\lambda, \forall\, \lambda \in \Lambda$,* id est, *there is a topology which is the least upper bound of $\{\tau_\lambda\}_{\lambda\in\Lambda}$;*

ii. *there exists the finest topology in the family of topologies coarser than $\tau_\lambda, \forall\, \lambda \in \Lambda$,* id est, *there is a topology which is the greatest lower bound of $\{\tau_\lambda\}_{\lambda\in\Lambda}$.*

Least upper bound and greatest lower bound should be understood, in both cases, with respect to the inclusion order. □

Proof:

i. Let $\mathcal{F}$ be the family of all topologies finer than $\tau_\lambda, \forall\, \lambda \in \Lambda$. Due to the Axiom Schema of Separation this is indeed a set*, since

$$\mathcal{F} = \{\tau \in \mathbb{P}(\mathbb{P}(X)); \tau \text{ is a topology finer than } \tau_\lambda, \forall\, \lambda \in \Lambda\}. \tag{2.3}$$

Due to Theorem 12, we know that $\tau = \bigcap \mathcal{F}$ is a topology as well. Notice that $\tau \subseteq \tau_{\mathcal{F}}, \forall\, \tau_{\mathcal{F}} \in \mathcal{F}$. Furthermore, since $\tau_\lambda \subseteq \tau_{\mathcal{F}}, \forall\, \lambda \in \Lambda, \forall\, \tau_{\mathcal{F}} \in \mathcal{F}$, it follows that $\tau_\lambda \subseteq \tau, \forall\, \lambda \in \Lambda$. Thus, $\tau \in \mathcal{F}$. Given that we already know that $\tau \subseteq \tau_{\mathcal{F}}, \forall\, \tau_{\mathcal{F}} \in \mathcal{F}$, we conclude τ is the coarsest topology in the family of topologies finer than $\tau_\lambda, \forall\, \lambda \in \Lambda$.

ii. Let $\mathcal{C}$ be the family of all topologies coarser than $\tau_\lambda, \forall\, \lambda \in \Lambda$. An argument similar to the one used in the previous item proves the fact that $\mathcal{C}$ is a set. However, this time let us define $\tau = \bigcap_{\lambda\in\Lambda} \tau_\lambda$. Notice that $\tau \subseteq \tau_\lambda, \forall\, \lambda \in \Lambda$. Therefore, $\tau \in \mathcal{C}$.

Let $\tau' \in \mathcal{C}$. By definition of $\mathcal{C}$, $\tau' \subseteq \tau_\lambda, \forall\, \lambda \in \Lambda$. Therefore, $\tau' \subseteq \bigcap_{\lambda\in\Lambda} \tau_\lambda = \tau$. Since $\tau' \subseteq \tau, \forall\, \tau' \in \mathcal{C}$, it is proved that τ is the finest topology in the family of topologies coarser than $\tau_\lambda, \forall\, \lambda \in \Lambda$. ■

*Perhaps you are not bored about whether this is or not a set, and in this case you probably are not interested on this footnote. Otherwise, you might be interested in Axiomatic Set Theory and might want to have a look at references [5, 12, 14].

Scholium:

You might realize that nowhere in the proof of Proposition 13 i. we used the hypothesis that $\{\tau_\lambda\}_{\lambda\in\Lambda}$ is a family of topologies. We could simply say that a topology τ is finer than an arbitrary set τ_λ whenever $\tau_\lambda \subseteq \tau$ and the argument would still hold. Thus, we *can* define the coarsest topology that makes a predefined collection of sets be a collection of open sets. ♣

This is our first example of how we can specify a topology by defining a smaller set, instead of the topology as a whole. You might have noticed that every topology we have shown could be explicitly written (except for those which were intersections of families of topologies). However, it is not exactly trivial to write the explicit form of a metric topology, for example. This is only one example of a case in which it is easier for us to specify a small set that can be used to describe the topology as a whole (for metric spaces, it is enough to specify the metric, and therefore which are the open balls). How could we obtain a wider sense of the collection of open balls in a metric sense? Can we define a "basis" for a topology?

When we were dealing with metric spaces, we defined an open set to be a such that had all points as interior points, *id est*, every point of that set could be "covered" by an open ball that was contained within said set.

In order to make things more clear, suppose (X, τ) is a topological space and we want to describe τ as if it was generated by some weird collection of "open balls" which is not necessarily associated to a metric. Let $\mathfrak{B}$ denote this collection. Since $X \in \tau$ and we want every point of X to be covered by some "open ball", we must require that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

As a second requirement, we are going to ask that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \exists\, \mathcal{B}_3 \in \mathfrak{B}; \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, which does hold for open balls in metric spaces, though the reason it is so important is not so clear right now, but it should be more evident within some time.

Let us then define a basis for a topology:

Definition 14 [Basis for a Topology on a Set]**:**

Let X be a set. We say $\mathfrak{B} \subseteq \mathbb{P}(X)$ is a *basis* for a topology on X whenever the following conditions hold:

i. $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$;

ii. $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. ♠

We must then check whether we can or not generate a topology using this concept. When dealing with open balls, a set would be open whenever all of its points were interior points. If we denote the collection of open balls as $\mathfrak{B}$, it means the metric topology is

$$\tau := \{O \subseteq X \,|\, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\}\,. \tag{2.4}$$

Indeed, the same is going to be true in our wider context.

Theorem 15:

Let X be a set and $\mathfrak{B}$ be a basis for a topology on X. Then the collection τ defined as

$$\tau := \{O \subseteq X \,|\, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\} \tag{2.5}$$

defines a topology on X. □

Proof:

The fact that $\varnothing \in \tau$ is vacuously true. $X \in \tau$ as well, since $\mathcal{B} \subseteq X, \forall \mathcal{B} \in \mathfrak{B}$ and $\forall x \in X, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

Given $O_1, O_2 \in \tau$, $O_1 \cap O_2 \in \tau$. Indeed, $\forall x \in O_1 \cap O_2$ we know there are $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ such that $x \in \mathcal{B}_1 \subseteq O_1$ and $x \in \mathcal{B}_2 \subseteq O_2$, since $O_1, O_2 \in \tau$. Therefore, $x \in \mathcal{B}_1 \cap \mathcal{B}_2$ and we know that $\exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, since $\mathfrak{B}$ is a basis. Since $\mathcal{B}_1 \subseteq O_1$ and $\mathcal{B}_2 \subseteq O_2$, it follows that $x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \subseteq O_1 \cap O_2$ and thus $\exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq O_1 \cap O_2$, which proves that $O_1 \cap O_2 \in \tau$.

Finally, let Λ be an arbitrary set of indexes and $O_\lambda \in \tau, \forall \lambda \in \Lambda$. Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Then, for any $x \in O$, there must be $\mathcal{B} \in \mathfrak{B}$ such that $x \in \mathcal{B} \subseteq O$, because $x \in O_\lambda$ for some $\lambda \in \Lambda$, and, since $O_\lambda \in \tau$, there is $\mathcal{B} \in \mathfrak{B}$ such that $x \in \mathcal{B} \subseteq O_\lambda$. We know that $O_\lambda \subseteq O$, and thus the theorem is proved. ■

Notice that the proof of Theorem 15 exhibits the importance of the second requirement made when defining a basis.

There is still another way of describing the topology generated by a basis. Perhaps you recall from real analysis that a set on $\mathbb{R}$ is open if, and only if, it can be written as unions of open intervals (which are nothing but the open balls in $\mathbb{R}$ with the usual metric). A similar result holds in this much more general context.

Lemma 16:

Let X *be a set and let* $\mathfrak{B}$ *be a basis for a topology on* X. *Then the topology* τ *generated by* $\mathfrak{B}$ *is the collection of all the sets of* $\mathbb{P}(X)$ *that can be written as unions of elements of* $\mathfrak{B}$. □

Proof:

Let υ be the collection of all the sets of $\mathbb{P}(X)$ that can be written as unions of elements of $\mathfrak{B}$. We what to prove that $\tau = \upsilon$. As usual, we will prove that $\upsilon \subseteq \tau$ and $\tau \subseteq \upsilon$.

The first inclusion is easy. Let $\mathcal{B} \in \mathfrak{B}$. $\forall x \in \mathcal{B}, \exists \mathcal{B}' = \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}' = \mathcal{B} \subseteq \mathcal{B}$. Therefore, $\mathfrak{B} \subseteq \tau$. Since τ is closed under arbitrary unions, of course $\upsilon \subseteq \tau$.

Let $O \in \tau$. We know that $\forall x \in O, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$. Let $\mathfrak{B}_O$ be the collection of such sets $\mathcal{B}$. Then notice that, $\forall x \in O$, $x \in \bigcup \mathfrak{B}_O$, and thus $O \subseteq \bigcup \mathfrak{B}_O$. Since $\mathcal{B} \subseteq O, \forall \mathcal{B} \in \mathfrak{B}_O$, we know that $x \in O, \forall x \in \bigcup \mathfrak{B}_O$. Thus, $\bigcup \mathfrak{B}_O \subseteq O$. It follows that $O = \bigcup \mathfrak{B}_O$. Since the latter is nothing but a union of elements in $\mathfrak{B}$, we have proved that $\tau \subseteq \upsilon$, and the lemma follows. ■

Example [Product Topology]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then $X \times Y$ can be turned into a topological space the topology generated by the basis $\mathfrak{B} = \{O_X \times O_Y; O_X \in \tau_X, O_Y \in \tau_Y\}$. This is one of the ways of defining the so called *product topology* (we shall see another one in Definition 68).

We must proceed to check that $\mathfrak{B}$ is indeed a basis. Let $(x, y) \in X \times Y$. Since τ_X and τ_Y are topologies, we know that $X \in \tau_X$ and $Y \in \tau_Y$, and thus $X \times Y \in \mathfrak{B}$. Therefore, $\forall (x, y) \in X \times Y, \exists \mathcal{B} \in \mathfrak{B}; (x, y) \in \mathcal{B}$.

Let then $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ and $(x, y) \in \mathcal{B}_1 \cap \mathcal{B}_2$. Since $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, we know there are $O_i^X \in \tau_X$ and $O_i^Y \in \tau_Y, i = 1, 2$, such that $\mathcal{B}_1 = O_1^X \times O_1^Y$, with a similar relation for $\mathcal{B}_2$.

Thus, $(x, y) \in (O_1^X \times O_1^Y) \cap (O_2^X \times O_2^Y) = (O_1^X \cap O_2^X) \times (O_1^Y \cap O_2^Y)$. Since τ_X is a topology, $O_1^X \cap O_2^X \in \tau_X$, with a similar result for Y. Therefore, $\mathcal{B} = (O_1^X \cap O_2^X) \times (O_1^Y \cap O_2^Y) \in \mathfrak{B}$. It follows that $(x, y) \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. This finishes the proof that $\mathfrak{B}$ is indeed a basis for a topology on $X \times Y$. ♥

We might yet wonder whether we can start with a topological space and find a basis for the topology we are dealing with. In other words, we can use a basis to obtain a topology, but what about going the other way around? Does every topology admit a basis?

Proposition 17:

Let (X, τ) *be a topological space and* $\mathfrak{B} \subseteq \tau$ *be a collection of sets such that* $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$. *Then* $\mathfrak{B}$ *is a basis for a topology on* X, *with* τ *being generated by* $\mathfrak{B}$. □

Proof:

Firstly, we want to prove that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$. This follows directly from the fact that $X \in \tau$, for τ is a topology. Since $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in B \subseteq O$, we simply set $O = X$ and we get that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

We must then prove that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$. Since $\mathfrak{B} \subseteq \tau$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are open sets and so is their intersection. Thus, since $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$, we might set $O = \mathcal{B}_1 \cap \mathcal{B}_2$ and obtain $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, as desired. ■

As we noted earlier, it might be easier to deal with bases instead of topologies. It is a natural conclusion that it would be interesting if we could compare two comparable topologies by simply taking a look at the bases that generate them.

Proposition 18:

Let X *be a set,* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be each a basis for a topology on* X *and let* τ *and* τ' *be the topologies generated by* $\mathfrak{B}$ *and* $\mathfrak{B}'$*, respectively. Then the following affirmations are equivalent:*

i. τ' *is finer than* τ;

ii. $\forall\, x \in X, \forall\, \mathcal{B} \in \mathfrak{B}$ *with* $x \in \mathcal{B}, \exists\, \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$. □

Proof:

i ⇒ ii: If τ' is finer than τ, then $\tau \subseteq \tau'$. Therefore, ii surely holds, because $\forall\, \mathcal{B} \in \mathfrak{B}, \mathcal{B} \in \mathfrak{B}'$.

ii ⇒ i: If $\forall\, x \in X, \forall\, \mathcal{B} \in \mathfrak{B}$ with $x \in \mathcal{B}, \exists\, \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$, then $\mathcal{B}$ is in the topology generated by $\mathfrak{B}'$ (as you can see from Theorem 15). Therefore, $\mathfrak{B} \subseteq \tau'$. Since a topology is closed under arbitrary unions and τ is nothing but the collection of all sets that can be written as unions of elements of $\mathfrak{B}$ (Lemma 16), it follows that $\tau \subseteq \tau'$, *id est*, τ' is finer than τ. ■

One might wonder why would we even care about which topology is finer or coarser. Is this nomenclature really useful? Well, the usefulness of a name is certainly always questionable. As Shakespeare would put, "What's in a name? That which we call a rose/By any other name would smell as sweet." (Romeo and Juliet (Act II, Scene ii, 45-46)). Nevertheless, the existence of a name for this concept will be useful when we deal with some more topological properties on metric spaces.

Since I've been saying "the usual topology on $\mathbb{R}$" in a sloppy way for some time, perhaps we should define some names for the main topologies we use on $\mathbb{R}$.

Definition 19 [Common Topologies on $\mathbb{R}$]**:**

Consider the real line $\mathbb{R}$.

We define the *standard topology* on $\mathbb{R}$ as the topology generated by the open intervals ($\mathfrak{B} = \{(a, b) ; a, b \in \mathbb{R}\}$). Unless specified otherwise, every time we mention $\mathbb{R}$ it should be understood that $\mathbb{R}$ comes along with the standard topology. This is the topology I meant (and still mean) by *usual topology*.

We define the *Sorgenfrey topology*, also known as the *lower-limit topology*, on $\mathbb{R}$ as the topology generated by the set $\mathfrak{B}' = \{[a, b) ; a, b \in \mathbb{R}\}$. The topological space formed by $\mathbb{R}$ with the Sorgenfrey topology is eventually called Sorgenfrey line.

Finally, we let $K = \left\{\frac{1}{n}; n \in \mathbb{N}^*\right\}$. The K-*topology* on $\mathbb{R}$ is the topology generated by the basis $\mathfrak{B}'' = \{(a, b) ; a, b \in \mathbb{R}\} \cup \{(a, b) \backslash K; a, b \in \mathbb{R}\}$. ♠

This time, I will leave you the joy of proving the claim that $\mathfrak{B}, \mathfrak{B}'$ and $\mathfrak{B}''$ are indeed bases.

Examples [A Countable Basis for the Standard Topology on $\mathbb{R}$]**:**

Consider $\mathbb{R}$ along with its standard topology. It in fact admits a countable basis, namely*, the collection of open intervals centered at rational numbers with radius of the form $\frac{1}{n}, n \in \mathbb{N}$. We write $\mathfrak{B} = \left\{\mathcal{B}_{\frac{1}{n}}(x); x \in \mathbb{Q}, n \in \mathbb{N}\right\}$.

Firstly, we want to prove that $\forall x \in \mathbb{R}, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$. If $x \in \mathbb{Q}$, we might simply pick $n = 1$ and have $\mathcal{B}_1(x)$, which clearly contains x. On the other hand, if $x \in \mathbb{R}\backslash\mathbb{Q}$, then pick an arbitrary natural number n. Consider the interval $\mathcal{B}_{\frac{1}{n}}(x)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is at least one rational number $r \in \mathcal{B}_{\frac{1}{n}}(x)$. Since $|r - x| < \frac{1}{n}$, because $r \in \mathcal{B}_{\frac{1}{n}}(x)$, it follows that $x \in \mathcal{B}_{\frac{1}{n}}(r)$.

Secondly, let $x \in \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$, for some given $x_1, x_2 \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Since $x \in \mathcal{B}_{\frac{1}{n}}(x_1)$, $\frac{1}{n} > |x - x_1|$, with a similar result involving m and x_2. Let

$$r = \max\left\{\frac{1}{n} - |x - x_1|, \frac{1}{m} - |x - x_2|\right\}.$$

Due to the Archimedean property, we know that there is a natural number $p \in \mathbb{N}$ such that $\frac{1}{p} < r$. I leave to you the task of proving that $\mathcal{B}_r(x) \subseteq \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$ (suggestion: use the triangle inequality). Notice then that $\mathcal{B}_{\frac{1}{p}}(x) \subseteq \mathcal{B}_r(x) \subseteq \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$, proving that $\mathfrak{B}$ is a basis.

Finally, we must yet prove that $\mathfrak{B}$ is countable. Let f be the function $f\colon \mathbb{Q} \times \mathbb{N} \to \tau$ that maps $(r, n) \to f(r, n) = \mathcal{B}_{\frac{1}{n}}(r)$. τ denotes the usual topology on $\mathbb{R}$. If we restrict the range of f to its image, which is precisely $\mathfrak{B}$, we obtain a surjective function from $\mathbb{Q} \times \mathbb{N}$ to $\mathfrak{B}$.

f is also a injective function. Let $(r_1, n_1) \neq (r_2, n_2)$, both being elements of $\mathbb{Q} \times \mathbb{N}$. If $r_1 = r_2$, suppose without any loss of generality that $n_2 > n_1$. Since $r_1 + \frac{\frac{1}{n_1} - \frac{1}{n_2}}{2}$ is in $\mathcal{B}_{\frac{1}{n_1}}(r_1)$, but not in $\mathcal{B}_{\frac{1}{n_2}}(r_2)$. Thus, $f(r_1, n_1) \neq f(r_2, n_2)$.

*Another possibility would be picking intervals with rational extreme points.

Suppose now that $r_1 \neq r_2$, and suppose for contradiction that there are $n_1, n_2 \in \mathbb{N}$ such that $\mathcal{B}_{\frac{1}{n_1}}(r_1) = \mathcal{B}_{\frac{1}{n_2}}(r_2)$. Notice that $r_i \pm n_i, i = 1, 2$ is an upper (lower) bound for $\mathcal{B}_{\frac{1}{n_i}}(r_i)$, which is thus limited. The least upper bound (greatest lower bound) property then guarantees that there are $a, b \in \mathbb{R}$ such that $\mathcal{B}_{\frac{1}{n_i}}(r_i) = (a, b)$. I leave as an exercise to prove that in fact $b = r_i + n_i$ and $a = r_i - n_i$. It follows then that $r_1 = \frac{a+b}{2} = r_2$, contradicting our initial hypotheses that $r_1 \neq r_2$. It is then necessary that $f(r_1, n_1) \neq f(r_2, n_2)$, proving that f is injective.

We have proven that f is injective and surjective, and thus bijective. Since $\mathbb{Q} \times \mathbb{N}$ is the Cartesian product of two countable sets, it is countable itself. Since there is a bijection between $\mathfrak{B}$ and an countable set, $\mathfrak{B}$ is countable, *quo erat demonstrandum*. ♥

Proposition 20:

Consider the real line, $\mathbb{R}$. Sorgenfrey's topology and the K-topology are strictly finer than the standard topology, though they are not comparable to one another. □

Proof:

It is easy to see that the K-topology is finer than the standard topology: notice that $\mathfrak{B} \subseteq \mathfrak{B}$. Thus, of course $\forall x \in \mathbb{R}, \forall \mathcal{B} \in \mathfrak{B}$ with $x \in \mathcal{B}, \exists \mathcal{B}'' \in \mathfrak{B}''; x \in \mathfrak{B}'' \subseteq \mathfrak{B}$. Namely, $\mathcal{B}'' = \mathcal{B}$.

As for the Sorgenfrey line, let $x \in \mathbb{R}$ and let $a, b \in \mathbb{R}; a < x < b$, so that $x \in (a, b)$. Let $c = \frac{a+x}{2}$. Then $a < c < x$ and $x \in [c, b) \subseteq (a, b)$. Therefore, we've shown that $\forall x \in \mathbb{R}, \forall \mathcal{B} \in \mathfrak{B}$ with $x \in \mathfrak{B}, \exists \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$.

Proposition 18 ensures that both the Sorgenfrey topology and the K-topology are finer than the standard topology. We still have to prove that they are *strictly* finer than the standard topology and not comparable with each other.

Let $a, b \in \mathbb{R}; a < 0 < b$. Then $0 \in (a, b) \setminus K$. Notice that this is an open set for the K-topology. Assume there are $c, d \in \mathbb{R}$ such that $0 \in (c, d) \subseteq (a, b) \setminus K$. Then there are no elements in (c, d) with the form $\frac{1}{n}$. Since $0 \in (c, d)$, this means that $\exists\, d \in \mathbb{R}; \forall\, n \in \mathbb{N}^*, \frac{1}{n} \notin [0, d)$. However, since the Archimedean property holds in $\mathbb{R}$, $\forall\, d \in \mathbb{R}^*_+, \exists\, n \in \mathbb{N}^*; 0 < \frac{1}{n} < d$. Thus, we have reached a contradiction. Therefore, and due to Proposition 18, the standard topology cannot be finer than the K-topology, and thus the K-topology is strictly finer than the standard topology.

Let $x, a \in \mathbb{R}, a > x$. Then $x \in [x, a)$, which is an open set for the Sorgenfrey topology. Suppose that there are $b, c \in \mathbb{R}$ such that $x \in (b, c) \subseteq [x, a)$. If $x \in (b, c)$, then $b < x$. However, since $(b, c) \subseteq [x, a)$, $x \leqslant b$. Since it is impossible for $b < x$ and $x \leqslant b$ to be true simultaneously, we have reached a contradiction. Due to Proposition 18, it is not possible for the standard topology to be finer than the Sorgenfrey topology, and thus the latter is strictly finer than the former.

We can show that the Sorgenfrey topology is not finer than the K-topology in an analogous way to how we have shown that the standard topology is not finer than the K-topology. We can show that the K-topology is not finer than the Sorgenfrey topology in analogous way to how we have shown that the standard topology is not finer than the Sorgenfrey topology. Thus, since neither the Sorgenfrey topology nor the K-topology is finer than the other, they are not comparable. ■

We have been using only one of the axioms of a topology so far to specify a whole topology from a smaller set (the basis). Namely, the property that topologies are closed under arbitrary unions and we might write any open set as unions of elements from the base. One might then think whether we can generate a topology using the property that the topology is closed under finite intersections?

Definition 21 [Subbasis]**:**

Let X be a set. We say $\mathfrak{S}$ is a *subbasis* for a topology on X if the union of all the elements of $\mathfrak{S}$ equals X. The *topology generated by the subbasis* $\mathfrak{S}$ is the collection of unions of finite intersections of elements of $\mathfrak{S}$, *id est*, the topology generated by the basis $\mathfrak{B} = \{\bigcap_{i=1}^{n} \mathcal{B}_i; \mathcal{B}_i \in \mathfrak{S}\}$. ♠

Proposition 22:

The topology generated by a subbasis is indeed a topology. □

Proof:

Let X be a set and $\mathfrak{S}$ be a subbasis for a topology on X. Notice that our claim is equivalent to proving that $\mathfrak{B} = \{\bigcap_{i=1}^{n} \mathcal{B}_i; \mathcal{B}_i \in \mathfrak{S}\}$ is indeed a basis for a topology on X.

Firstly, we need to prove that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}, n \in \mathbb{N}^*$. Let $x \in X$. Since $\bigcup \mathfrak{S} = X$, we know that $\exists\, \mathcal{B} \in \mathfrak{S}; x \in \mathcal{B}$. Let $\mathcal{B}_x$ be such a set. Since $\bigcap_{i=1} 1\mathcal{B}_x = \mathcal{B}_x$, we know that $\mathcal{B}_x \in \mathfrak{B}$. Therefore, we have found an element of the basis $\mathfrak{B}$ which contains x.

We then proceed to prove that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ and let $x \in \mathcal{B}_1 \cap \mathcal{B}_2$. We know that there are sets $\mathcal{B}_1^i$ and $\mathcal{B}_2^i$ which are elements of $\mathfrak{S}$ such that $\mathcal{B}_1 = \bigcap_{i=1}^{n} \mathcal{B}_1^i$ and $\mathcal{B}_2 = \bigcap_{i=1}^{m} \mathcal{B}_2^i$. We know that $\left(\bigcap_{i=1}^{n} \mathcal{B}_1^i\right) \cap \left(\bigcap_{i=1}^{m} \mathcal{B}_2^i\right) = \mathcal{B} \in \mathfrak{B}$, since it is composed of finite intersections of elements of the subbasis. Notice now that $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$ and thus it holds that $x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Therefore, $\mathfrak{B}$ is indeed a basis for a topology on X and the topology generated by a subbasis is indeed a topology. ■

Finally, it is common within Mathematics to consider a substructure within a larger set. For example, one might speak about a linear subspace within a linear space. Therefore, it is natural to wonder if there is any topological structure a topological space (X, τ) could induce on a subset $Y \subseteq X$.

Definition 23 [Subspace of a Topological Space]**:**

Let (X, τ) be a topological space and let $Y \subseteq X$. We define the *subspace topology on* Y, also known as the *relative topology*, as the collection

$$\tau_Y \coloneqq \{O \cap Y; O \in \tau\}. \tag{2.6}$$

(Y, τ_Y) is then said to be a *subspace* of (X, τ). ♠

Perhaps you remember we have already proven that τ_Y is indeed a topology on Y. Have a look at the examples of topological spaces at page 10.

We already know that working with bases is easier than with the topologies themselves, and thus the following Lemma might prove itself useful.

Lemma 24:

Let (X, τ) be a topological space, $\mathfrak{B}$ be a basis for τ and $Y \subseteq X$. Then $\mathfrak{B}_Y \equiv \{\mathcal{B} \cap Y; \mathcal{B} \in \mathfrak{B}\}$ is a basis for the subspace topology in Y. □

Proof:

Let τ_Y denote the subspace topology on Y.

We know that $\mathcal{B} \in \tau, \forall \mathcal{B} \in \mathfrak{B}$ (this follows from Lemma 16). Thus, given $\mathcal{B} \in \mathfrak{B}$, we know that $\mathcal{B} \cap Y \in \tau_Y$, by the very definition of τ_Y. This implies $\mathfrak{B}_Y \subseteq \tau_Y$.

Due to Proposition 17, we only have to prove that $\forall U \in \tau_Y, \forall y \in U, \exists \mathcal{B}_Y \in \mathfrak{B}_Y; y \in \mathcal{B}_Y \subseteq U$. Therefore, let $U \in \tau_Y$ and consider some set $O \in \tau; O \cap Y = U$ (the existence of such a set is guaranteed by the definition of τ_Y). Let $y \in U \subseteq O$. Since $\mathfrak{B}$ is a basis for τ, we know that $\exists \mathcal{B} \in \mathfrak{B}; y \in \mathcal{B} \subseteq O$. Since $y \in U \subseteq Y$, we know that $y \in \mathcal{B} \cap Y = \mathcal{B}_Y$. Furthermore, notice that $\mathcal{B}_Y = \mathcal{B} \cap Y \subseteq O \cap Y = U$. Thus, $y \in \mathcal{B}_Y \subseteq U$. Notice that $\mathcal{B}_Y = \mathcal{B} \cap Y \in \mathfrak{B}_Y$ by the definition of $\mathfrak{B}_Y$.

Since the argument holds for every $U \in \tau_Y$ and for every $y \in U$, we have proven that $\mathfrak{B}_Y$ is indeed a basis for the subspace topology. ■

Remark:

Notice that I avoided saying U is open or O is open in the previous proof. Instead, I preferred saying $U \in \tau_Y$ or $O \in \tau$. When dealing with subspaces, we find a glitch with our current nomenclature, for saying O is open is too ambiguous. Therefore, it is usual for us to say that U is open in Y or O is open in X. Or even U is open *relative* to Y. Beware: not every set that is open in Y is also open in X. ♣

Example [A Set That is Open in a Subspace Only]**:**

Let $X = \{a, b, c\}$ and consider the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Consider $Y = \{a, b\} \subseteq X$ as a subspace of (X, τ). Notice that the subspace topology τ_Y is given by

$$\begin{aligned} \tau_Y &= \{\emptyset \cap Y, X \cap Y, \{a\} \cap Y, \{b, c\} \cap Y\}, \\ &= \{\emptyset, Y, \{a\}, \{b\}\}. \end{aligned} \tag{2.7}$$

The sets $Y = \{a, b\}$ and $\{b\}$ are open in Y, but not in X. ♥

Okay, that is kind of a bummer. It would certainly be interesting if we could have $\tau_Y \subseteq \tau$. This might not be the general situation, but it is possible if we admit another assumption: $Y \in \tau$.

Lemma 25:

Let (X, τ) be a topological space. Let $Y \subseteq X$ and let τ_Y be the subspace topology on Y. If Y is open in X, then $\tau_Y \subseteq \tau$. □

Proof:

Since $\tau_Y = \{O \cap Y; O \in \tau\}$, we simply want to prove that $O \cap Y \in \tau, \forall O \in \tau$. Since $Y \in \tau$ and topologies are closed under finite intersection, it is guaranteed that $O \cap Y \in \tau$. ■

3 The Road to Limits: Closures and Closed Sets

As suggests the name of this section, the next step we must take in our journey is understanding what is a closed set. It will become clear that there is a strong connection between the concept of a limit point and of a closed set, just as seen, for example, in Real Analysis and Metric Spaces. Besides, if some sets are open, I guess it makes some sense for closed sets to exist.

Definition 26 [Closed Set]**:**

Let (X, τ) be a topological space. Se say a set $A \subseteq X$ is a *closed set* if, and only if, $A^c \in \tau$, *id est*, whenever the complement of A is an open set. ♠

However, there is a huge difference between sets and doors: a door must be either open or closed, but that is not true for sets. Sets might be open, closed, both or neither.

Example:

Let (X, τ) be a topological space. X and $\varnothing$ are both open and closed sets, because $\varnothing^c = X \in \tau$ and $X^c = \varnothing \in \tau$.

Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$. $\{a\}$ and $\{b\}$ are both open and closed (because one is the complement of the other and vice-versa). Nevertheless, $\{a, b\} \subseteq X$ isn't open nor closed, for $\{a, b\} \notin \tau$ and $\{a, b\}^c = \{c\} \notin \tau$. ♥

Once more, notice that we have a problem with nomenclature when dealing with subspaces. Thus, we are also going to say closed in Y or closed relative to Y, and so on.

Proposition 27:

Let (X, τ) be a topological space. Let $Y \subseteq X$ and let τ_Y be the subspace topology on Y. $A \subseteq Y$ is closed in Y if, and only if, $A = F \cap Y$ for some closed set $F \subseteq X$. □

Proof:

$\Leftarrow$: Assume $A = F \cap Y$ for some closed set $F \subseteq X$. Notice that

$$\begin{aligned} Y \backslash A &= Y \cap A^c, \\ &= Y \cap (F \cap Y)^c, \\ &= Y \cap (F^c \cup Y^c), \\ &= Y \cap F^c. \end{aligned} \tag{3.1}$$

Since F is closed in X, F^c is open in X, and it follows that $Y \backslash A$ is open in Y. Therefore, A is closed in Y.

$\Rightarrow$: Assume A is closed in Y. Then there is some set $O \in \tau$ such that $Y \backslash A = Y \cap O$. Therefore,

$$\begin{aligned} A &= Y \cap (Y \cap O)^c, \\ &= Y \cap (Y^c \cup O^c), \\ &= Y \cap O^c. \end{aligned} \tag{3.2}$$

As O is closed in X, the result is proved. ■

Corollary 28:

Let (X, τ) be a topological space. Let $Y \subseteq X$ be a closed set and let τ_Y be the subspace topology on Y. $A \subseteq Y$ is closed in Y if, and only if, it is closed in X. □

Proof:

Due to Proposition 27 we know that A is closed in Y if, and only if, $A = F \cap Y$ for some closed set in X. Since Y is closed in X, $F \cap Y$ is closed in X. Thus, A is closed in Y if, and only if, it is closed in X. ■

Definition 29 [Clopen Sets]**:**

Let (X, τ) be a topological space. We say a set $A \subseteq X$ is a *clopen set* whenever both A and A^c are open sets, *id est*, whenever A is both closed and open. ♠

An interesting reason for us to study closed sets is because we could also choose to define topological spaces through the structure of the closed sets, instead of the structure of open sets.

Theorem 30:

Let (X, τ) *be a topological space. Let* $\phi = \{F \in \mathbb{P}(X); F^c \in \tau\}$. *Notice that* ϕ *is the set of all closed sets in* X. *Finally, let* Λ *be an arbitrary set of indexes.* ϕ *satisfies the following properties:*

i. $\varnothing, X \in \phi$;

ii. $A, B \in \phi \Rightarrow A \cup B \in \phi$;

iii. $A_\lambda \in \phi, \forall \lambda \in \Lambda \Rightarrow \bigcap_{\lambda \in \Lambda} A_\lambda \in \phi$. □

Proof:

i. Since $\varnothing, X \in \tau$ and they are the complements of each other, $\varnothing, X \in \phi$;

ii. Given that $A^c, B^c \in \tau$, we know that $(A^c \cap B^c) \in \tau$. Thus, $(A^c \cap B^c)^c = A \cup B \in \phi$.

iii. Given that ${A_\lambda}^c \in \tau, \forall \lambda \in \Lambda$, we know that $\bigcup_{\lambda \in \Lambda} {A_\lambda}^c \in \tau$. Therefore,

$$\left(\bigcup_{\lambda \in \Lambda} {A_\lambda}^c\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda \in \phi. \tag{3.3}$$

This concludes the proof. ■

Although unusual, one could define a topological space as a set X with a collection ϕ of closed sets obeying the properties of Theorem 30, define open sets as complements of closed sets and reobtain the results we have already found.

We shall soon see as well that it is interesting that, given a set, we may obtain an open set, or a closed set related to that set. It is natural to define such generated sets through the properties we already know about intersections of closed sets and unions of open sets.

Definition 31 [Interior, Closure and Boundary]**:**

Let (X, τ) be a topological space and let $A \subseteq X$. We define the *interior of* A, denoted $\mathring{A}$ (or, equivalently, $\operatorname{int} A$), as the union of all open sets contained within A. We define the *closure of* A, denoted $\overline{A}$, as the intersection of all closed sets containing A. Finally, we define the boundary of A, ∂A, as $\partial A = \overline{A} \backslash \mathring{A}$. ♠

Proposition 32:

Let (X, τ) *be a topological space and* $A \subseteq X$. *Then the following results hold:*

i. $\mathring{A} \subseteq A \subseteq \overline{A}$;

ii. $\mathring{A} = A$ *if, and only if,* A *is open;*

iii. $\overline{A} = A$ *if, and only if,* A *is closed.*

□

Proof:

i. since $\mathring{A}$ is the union of every open set contained within A, it surely is contained within A, *id est*, $\mathring{A} \subseteq A$. Similarly, provided that $\overline{A}$ is the intersection of all closed sets containing A, it surely contains A, and thus $A \subseteq \overline{A}$.

ii. $\mathring{A}$ is an arbitrary union of open sets, and thus is open. Therefore, if $\mathring{A} = A$, A certainly is open. On the other hand, we know that $\mathring{A} \subseteq A$. If A is open, then every point of A belongs in the union of every open set contained within A, and thus $A \subseteq \mathring{A}$. Therefore, if A is open, then $\mathring{A} = A$.

iii. $\overline{A}$ is an arbitrary intersection of closed sets, and thus is closed. Therefore, if $\overline{A} = A$, A certainly is closed. On the other hand, we know that $A \subseteq \overline{A}$. If A is closed, then every point of the intersection of all closed sets containing A is a point of A, *id est*, $\overline{A} \subseteq A$. Thus, if A is closed, then $\overline{A} = A$. ■

Lemma 33:

Let (X, τ) *be a topological space and let* $A \subseteq X$*. Then, on the inclusion order,* $\mathring{A}$ *is the greatest open set contained within* A *and* $\overline{A}$ *is the smallest closed set containing* A*.* □

Proof:

Since $\mathring{A}$ is the union of every open set contained in A, if any open set, say B, is larger than (or not comparable to) $\mathring{A}$ and still contained within A, then B would also be on the family whose union we are taking, and thus $B \subseteq \mathring{A}$. Since we have reached a contradiction, it follows that there is no such set B.

A similar reasoning holds for $\overline{A}$. Since $\overline{A}$ is the intersection of every closed set containing A, the existence of any closed set B containing A implies that B is on the family whose union we are taking. Therefore, $\overline{A} \subseteq B$. ■

We might then examine some properties of the operations that take some set to its closure, interior and/or boundary.

Proposition 34:

Let (X, τ) *be a topological space. Let* $A, B \subseteq X$*. Let* Λ *be an arbitrary family of indexes and* $A_\lambda \subseteq A, \forall \lambda \in \Lambda$*. Then the following properties hold:*

i. $\overline{\overline{A}} = \overline{A}$;

ii. $B \subseteq A \Rightarrow \overline{B} \subseteq \overline{A}$;

iii. $\overline{B \cup A} = \overline{B} \cup \overline{A}$;

iv. $\overline{\bigcap_{\lambda\in\Lambda} A_\lambda} \subseteq \bigcap_{\lambda\in\Lambda} \overline{A_\lambda}$;

v. $\overline{\varnothing} = \varnothing, \overline{X} = X$. □

Proof:

i. We know $\overline{A}$ is a closed set (Lemma 33), and thus Proposition 32 implies $\overline{A} = \overline{\overline{A}}$;

ii. Proposition 32 guarantees that $B \subseteq A \subseteq \overline{A}$. Provided that $\overline{B}$ is the smallest closed set containing B and $\overline{A}$ is a closed set containing B, it follows that $\overline{B} \subseteq \overline{A}$;

iii. We know $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, and therefore $A \cup B \subseteq \overline{A} \cup \overline{B}$. Both $\overline{A}$ and $\overline{B}$ are closed sets and the finite union of closed sets is closed, and thus it follows that $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, and therefore $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

   Furthermore, $A \subseteq A \cup B \subseteq \overline{A \cup B}$. We know $\overline{A}$ is the smallest closed set containing A, and therefore $\overline{A} \subseteq \overline{A \cup B}$. An analogous arguments applies to B. $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$, and hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. It has already been proved that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

iv. $A_\lambda \subseteq \overline{A_\lambda}, \forall \lambda \in \Lambda$. Therefore, $\bigcap_{\lambda\in\Lambda} A_\lambda \subseteq \bigcap_{\lambda\in\Lambda} \overline{A_\lambda}$. However, $\overline{\bigcap_{\lambda\in\Lambda} A_\lambda}$ is the smallest closed set containing $\bigcap_{\lambda\in\Lambda} A_\lambda$. It follows that $\overline{\bigcap_{\lambda\in\Lambda} A_\lambda} \subseteq \bigcap_{\lambda\in\Lambda} \overline{A_\lambda}$.

v. From Theorem 30 we know that $\varnothing$ and X are closed sets. The result is then a consequence of Proposition 32. ■

A very similar set of properties holds for the interior of a set.

Proposition 35:

Let (X, τ) be a topological space. Let $A, B \subseteq X$. Let Λ be an arbitrary family of indexes and $A_\lambda \subseteq A, \forall \lambda \in \Lambda$. Then the following properties hold:

i. $\mathring{\mathring{A}} = \mathring{A}$;

ii. $B \subseteq A \Rightarrow \mathring{B} \subseteq \mathring{A}$;

iii. $\mathrm{int}(B \cap A) = \mathring{B} \cap \mathring{A}$;

iv. $\bigcup_{\lambda\in\Lambda} \mathring{A}_\lambda \subseteq \mathrm{int}(\bigcup_{\lambda\in\Lambda} A_\lambda)$;

v. $\mathring{\varnothing} = \varnothing, \mathring{X} = X$. □

Proof:

i. Just as when dealing with closure, we know $\mathring{A}$ is open and thus equals its interior, *id est*, $\mathring{A} = \mathring{\mathring{A}}$;

ii. If $B \subseteq A$, we know $\mathring{B} \subseteq B \subseteq A$. As $\mathring{A}$ is the largest open set contained in A and $\mathring{B}$ is an open set contained in A, it follows that $\mathring{B} \subseteq \mathring{A}$;

iii. We know that $A \cap B \subseteq A$, and thus $\text{int}(A \cap B) \subseteq \mathring{A}$, with a similar result for B. Therefore, $\text{int}(A \cap B) \subseteq \mathring{A} \cap \mathring{B}$.

On the other hand, $\mathring{A} \subseteq A$ and $\mathring{B} \subseteq B$. Therefore, $\mathring{A} \cap \mathring{B} \subseteq A \cap B$. The finite intersection of open sets is open, and thus $\mathring{A} \cap \mathring{B}$ is an open set contained within $A \cap B$. As $\text{int}(A \cap B)$ is the largest open set contained within $A \cap B$, it follows that $\mathring{A} \cap \mathring{B} \subseteq \text{int}\{A \cap B\}$. Hence, $\mathring{A} \cap \mathring{B} = \text{int}\{A \cap B\}$.

iv. As $A_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda, \forall \lambda \in \Lambda$, it holds that $\mathring{A}_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda, \forall \lambda \in \Lambda$. Thus, $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$. Since the arbitrary union of open sets is open, $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda$ is an open set contained within $\bigcup_{\lambda \in \Lambda} A_\lambda$. However, the largest open set contained within $\bigcup_{\lambda \in \Lambda} A_\lambda$ is $\text{int}(\bigcup_{\lambda \in \Lambda} A_\lambda)$, and therefore $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda \subseteq \text{int}(\bigcup_{\lambda \in \Lambda} A_\lambda)$;

v. Both $\varnothing$ and X are open sets and thus are equal to their interiors. ■

We might then relate the concepts of closure and interior.

Lemma 36:
Let (X, τ) be a topological space and let $A \subseteq X$. Then $\mathring{A} = \left(\overline{(A^c)}\right)^c$, or, equivalently, $\overline{A} = \left((\mathring{A^c})\right)^c$. □

Proof:
Firstly, notice that both expressions are indeed equivalent: simply exchange $A \leftrightarrow A^c$ and use the fact that $(A^c)^c = A$.

Let us now focus on the actual result. $\left(\overline{(A^c)}\right)^c$ is the complement of a closed set, and therefore it is an open set. As $A^c \subseteq \overline{(A^c)}$, it holds that $\left(\overline{(A^c)}\right)^c \subseteq A$. We know, however, that $\mathring{A}$ is the largest open set contained in A, and thus $\left(\overline{(A^c)}\right)^c \subseteq \mathring{A}$.

Next, we want to prove that $\mathring{A} \subseteq \left(\overline{(A^c)}\right)^c$, which is the same as proving that $\overline{(A^c)} \subseteq \left(\mathring{A}\right)^c$.

Since $\mathring{A} \subseteq A$, $A^c \subseteq \left(\mathring{A}\right)^c$. Furthermore, since it is the complement of an open set, $\left(\mathring{A}\right)^c$ is closed, making it a closed set containing A^c. The smallest closed set containing A^c is $\overline{(A^c)}$, and thus $\overline{(A^c)} \subseteq \left(\mathring{A}\right)^c$, as desired. ■

We might as well find properties pertinent to the boundary of a set. However, it is going to be useful to prove a small lemma before we can actually study those properties.

Lemma 37:
Let (X, τ) be a topological space and $A \subseteq X$. Then $\partial A = \overline{A} \cap \overline{(A^c)}$. □

Proof:

$$\begin{aligned}\partial A &= \overline{A} \backslash \mathring{A}, \\ &= \overline{A} \cap \left(\mathring{A}\right)^c, \\ &= \overline{A} \cap \overline{(A^c)}.\end{aligned} \tag{3.4}$$

The last step is justified by Lemma 36. ■

Proposition 38:

Let (X, τ) *be a topological space and* $A \subseteq X$*. Then the following hold:*

i. $\overline{\partial A} = \partial A$;

ii. $\partial A = \partial (A^c)$;

iii. $\partial (\partial A) \subseteq \partial A$;

iv. $\partial (\partial A) = \partial A \Leftrightarrow \text{int}(\partial A) = \varnothing$;

v. $\partial (\partial (\partial A)) = \partial (\partial A)$;

vi. $\partial \varnothing = \varnothing$ *and* $\partial X = X$. □

Proof:

i. Due to Lemma 37 we know ∂A is the intersection of two closed sets, and thus is closed as well. Therefore, it coincides with its closure;

ii. The definition of boundary presented on Lemma 37 is symmetric in A and A^c, *id est*, we might interchange them with no difference. Thus, $\partial A = \partial (A^c)$;

iii. If B is a closed set, then $\partial B \subseteq B$, for $\partial B = \overline{B} \backslash \mathring{B} \subseteq \overline{B} = B$. Given that ∂A is always closed, the result follows;

iv. If $\text{int}(\partial A) = \varnothing$, then $\partial (\partial A) = \partial A \backslash \varnothing$, where we already used that ∂A is closed and has empty interior.

   On the other hand, if $\partial (\partial A) = \partial A$, then $\partial A = \partial A \backslash \text{int}(\partial A)$, *id est*, no element of ∂A is in $\text{int}\{\partial A\}$, and therefore the latter is the empty set;

v. We know that $\partial (\partial A) = \partial A \cap \overline{((\partial A)^c)}$, which may be rewritten as $\partial (\partial A) = \partial A \cap \left((\mathring{\partial A})\right)^c$, due to Lemma 36. Then Proposition 35 guarantees that $\text{int}(\partial (\partial A)) = \mathring{\partial A} \cap \text{int}\left(\left((\mathring{\partial A})\right)^c\right)$. As $\text{int}\left(\left((\mathring{\partial A})\right)^c\right) \subseteq \left((\mathring{\partial A})\right)^c$, it follows that $\text{int}(\partial (\partial A)) \subseteq \mathring{\partial A} \cap \left((\mathring{\partial A})\right)^c = \varnothing$. Thus, $\text{int}(\partial (\partial A)) = \varnothing$ and the result follows. ■

Something remarkably interesting about the concepts of interior and closure of a set is the possibility of giving yet another definition of topological space. Instead of providing a set with the collection of open (or closed) subsets, we may as well equip it with an unary operation with some properties satisfied by the closure operation.

Definition 39 [Kuratowski Operator]**:**

Let X be a non-empty set and let $\kappa\colon \mathbb{P}(X) \to \mathbb{P}(X)$ be a function. κ is said to be a *Kuratowski operator* if, and only if, it satisfies the *Kuratowski axioms*:

i. $\kappa(\varnothing) = \varnothing$;

ii. $A \subseteq \kappa(A), \forall A \in \mathbb{P}(X)$;

iii. $\kappa(\kappa(A)) = \kappa(A), \forall A \in \mathbb{P}(X)$;

iv. $\kappa(A \cup B) = \kappa(A) \cup \kappa(B), \forall A, B \in \mathbb{P}(X)$. ♠

Theorem 40:

Let X be a non-empty set and let $\kappa\colon \mathbb{P}(X) \to \mathbb{P}(X)$ be a Kuratowski operator. Let τ_κ be defined as

$$\tau_\kappa := \{O \in \mathbb{P}(X); \kappa(O^c) = O^c\}. \tag{3.5}$$

Under these assumptions, it holds that (X, τ_κ) is a topological space. Furthermore, the closure $\overline{A}$ of a set A according to τ_κ respects $\overline{A} = \kappa(A)$. □

Proof:

Let us first define $\phi_\kappa := \{F \in \mathbb{P}(X); \kappa(F) = F\}$. Let Λ be an arbitrary set of indexes. I claim that

i. $\varnothing, X \in \phi_\kappa$;

ii. $A, B \in \phi_\kappa \Rightarrow A \cup B \in \phi_\kappa$;

iii. $A_\lambda \in \phi_\kappa, \forall \lambda \in \Lambda \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \in \phi_\kappa$.

Since κ is a Kuratowski operator, it holds that $\kappa(\varnothing) = \varnothing$ by hypothesis, and therefore $\varnothing \in \phi_\kappa$.

We also know that $A \subseteq \kappa(A), \forall A \in \mathbb{P}(X)$. Thus, $X \subseteq \kappa(X)$. However, $\kappa(X) \in \mathbb{P}(X)$, and therefore $\kappa(X) \subseteq X$. It follows that $\kappa(X) = X$, implying that $X \in \phi_\kappa$.

Let now $A, B \in \phi_\kappa$, *id est*, $\kappa(A) = A$ and $\kappa(B) = B$. As κ is a Kuratowski operator, $\kappa(A \cup B) = \kappa(A) \cup \kappa(B), \forall A, B \in \mathbb{P}(X)$. Hence,

$$\begin{aligned} \kappa(A \cup B) &= \kappa(A) \cup \kappa(B), \\ &= A \cup B, \end{aligned} \tag{3.6}$$

proving that $A \cup B \in \phi_\kappa$.

Finally, we want to prove that

$$A_\lambda \in \phi_\kappa, \forall \lambda \in \Lambda \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \in \phi_\kappa,$$

id est,

$$\kappa(A_\lambda) = A_\lambda, \forall \lambda \in \Lambda \Rightarrow \kappa\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} A_\lambda.$$

We know that κ is a Kuratowski operator, and therefore $A \subseteq \kappa(A), \forall A \in \mathbb{P}(X)$ is a given. We only need to prove that $\kappa\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subseteq \bigcap_{\lambda\in\Lambda} A_\lambda$.

$\forall\, A, B \in \mathbb{P}(X)$, if holds that* $A = (A \cap B) \sqcup (A \cap B^c)$. Thus, if $B \subseteq A$, it holds that $A = B \sqcup (A \backslash B)$. As κ is a Kuratowski operator, it follows that $\kappa(A) = \kappa(B) \cup \kappa(A \backslash B) \supseteq \kappa(B)$. Thus, whenever $B \subseteq A$, it follows that $\kappa(B) \subseteq \kappa(A)$.

Let now $\lambda_0 \in \Lambda$. We know that $\bigcap_{\lambda \in \Lambda} A_\lambda \subseteq A_{\lambda_0}$, and thus it follows that $\kappa\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \subseteq \kappa(A_{\lambda_0}), \forall\, \lambda_0 \in \Lambda$. Hence,

$$\kappa\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \subseteq \bigcap_{\lambda_0 \in \Lambda} \kappa(A_{\lambda_0}) = \bigcap_{\lambda \in \Lambda} A_\lambda, \tag{3.7}$$

for $\kappa(A_\lambda) = A_\lambda, \forall\, \lambda \in \Lambda$. Notice that as λ_0 is a dummy index, it can be changed to λ.

Thus, we already know that, given (X, κ), we may equip X with a collection of sets ϕ_κ satisfying the same properties closed sets have in topological spaces (Theorem 30). I claimed previously we could use such a space (X, ϕ_κ) to define a topological space and reobtain our usual definition of topology. Let us prove it.

Given ϕ_κ defined as before, we want to prove that the collection

$$\tau_\kappa \coloneqq \{O \in \mathbb{P}(X); O^c \in \phi_\kappa\} \tag{3.8}$$

is a topology in X. Notice that, currently, the κ indexes are merely aesthetic and the proof that a space with the notion of a closed set is a topological space is still completely general and independent of the Kuratowski operator.

Firstly, we want to prove that $\varnothing, X \in \tau_\kappa$. As $X, \varnothing \in \phi_\kappa$, $X^c = \varnothing \in \tau_\kappa$ and $\varnothing^c = X \in \tau_\kappa$.

Next, we want to prove that given $A, B \in \tau_\kappa$, $A \cap B \in \tau_\kappa$. As $A, B \in \tau_\kappa$, we know that $A^c, B^c \in \phi_\kappa$. Thus, $A^c \cup B^c \in \phi_\kappa$. Finally, $(A^c \cup B^c)^c = A \cap B \in \tau_\kappa$.

Finally, let Λ be an arbitrary set of indexes. Let $A_\lambda \in \tau_\kappa, \forall\, \lambda \in \Lambda$. We want to prove that $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau_\kappa$.

We have

$$\begin{gathered}
{A_\lambda}^c \in \phi_\kappa, \\
\bigcap_{\lambda \in \Lambda} {A_\lambda}^c \in \phi_\kappa, \\
\left(\bigcap_{\lambda \in \Lambda} {A_\lambda}^c\right)^c \in \tau_\kappa, \\
\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau_\kappa.
\end{gathered} \tag{3.9}$$

Thus, it is proven that (X, τ_κ) is a topological space. We also proved that given a space with a "closed topology", (X, ϕ), it can be regarded as a topological space.

Finally, we have to prove that $\overline{A} = \kappa(A), \forall\, A \in \mathbb{P}(X)$, with the closure considered with respect to the topology τ_κ.

Let $A \in \mathbb{P}(X)$. Then $\kappa(A)$ is a closed set. Indeed, $\kappa(A) = \kappa(\kappa(A))$ by hypothesis (for κ is a Kuratowski operator) and thus $\kappa(A) \in \phi_\kappa$. Let now F be a closed set such that $A \subseteq F$. Then we have that $F^c \in \tau_\kappa$ (for F is closed) and, thus, $F \in \phi_\kappa$. Therefore, it holds, from the definition of ϕ_κ, that $F = \kappa(F)$.

*We write $A \sqcup B$ to denote $A \cup B$, with $A \cap B = \varnothing$.

As $A \subseteq F$, it follows that $\kappa(A) \subseteq \kappa(F) = F$, *id est*, $\kappa(A) = F$. Hence, $\kappa(A)$ is the smallest closed set containing A, which by definition is the closure of A. Therefore, we have proved that $\overline{A} = \kappa(A)$. ■

Once more, we might have problems regarding what happens in subspaces. The following lemma exhibits how the closure in a subspace relates to the the closure on the topological space.

Lemma 41:

Let (X, τ) be a topological space, (Y, τ_Y) be a subspace, $A \subseteq Y$ and $\overline{A}$ be the closure of A in X. Then the closure of A in Y is given by $\overline{A} \cap Y$. □

Proof:

Due to Proposition 27, we know that $\overline{A} \cap Y$ is a closed set in Y. Suppose B is a closed set in Y such that $A \subseteq B \subseteq \overline{A} \cap Y$. Proposition 27 guarantees that there is a closed set D in X such that $B = D \cap Y$. Thus, $A \subseteq D \cap Y \subseteq \overline{A} \cap Y$. It follows that $A \subseteq D$ for some closed set D. However, since $\overline{A}$ is the intersection of every closed set containing A, we know that $\overline{A} \subseteq D$. Finally, we conclude that $\overline{A} \cap Y \subseteq B$, and therefore $\overline{A} \cap Y$ is the smallest closed set containing A, which coincides with the intersection of every closed set containing A (Lemma 33). ■

Even though we have proven a number of results on closures, interiors and boundaries, our current definition of closure is no good for making calculations. Given a set, our present knowledge concerning closures won't allow us to find the closure of such set in an easy way. We can, though, establish a definition of closure based on intersections of the given set with elements of a basis for the topology.

Definition 42 [Intersects]**:**

Let X be a set and $A, B \subseteq X$. We say A *intersects* B if, and only if, $A \cap B \neq \varnothing$. ♠

Theorem 43:

Let (X, τ) be a topological space and let $A \subseteq X$. Then it holds that

i. $x \in \overline{A}$ *if, and only if,* O *intersects* $A, \forall\, O \in \tau; x \in O$;

ii. *if* $\mathfrak{B}$ *is a basis for the topology* τ, *then* $x \in \overline{A}$ *if, and only if,* B *intersects* $A, \forall\, B \in \mathfrak{B}; x \in \mathfrak{B}$. □

Proof:

We shall do the proof by contrapositive, *id est*, we want to prove that $x \notin \overline{A} \Leftrightarrow \exists\, O \in \tau; x \in O, O \cap A = \varnothing$.

Suppose $x \notin \overline{A}$. Clearly $x \in \overline{A}^c$. By Lemma, we have that $x \in \mathrm{int}(A^c)$. We know that $\mathrm{int}(A^c)$ is an open set, by the very definition of interior. As $\mathrm{int}(A^c) \subseteq A^c$ and $A^c \cap A = \varnothing$, we have that $\mathrm{int}(A^c) \cap A = \varnothing$, proving the existence of a set $O \in \tau$ such that $x \in O, O \cap A = \varnothing$.

On the other hand, suppose $\exists\, O \in \tau; x \in O, O \cap A = \varnothing$. Since $O \cap A = \varnothing$, we know that $O \subseteq A^c$. Notice then that $A \subseteq O^c$ and that O^c is a closed set, for O is open by hypothesis. Therefore, $\overline{A} \subseteq O^c$, as $\overline{A}$ is the smallest closed set containing A. We have then that $\overline{A} \cap O = \varnothing$. As $x \in O$, by hypothesis, we have that $x \notin \overline{A}$, proving the result we wanted.

This proves the first item. We now must prove the second.

The first implication ($x \in \overline{A} \Rightarrow B \cap A \neq \emptyset, \forall B \in \mathfrak{B}; x \in B$) is simple: we already know the result is valid in general for any open set. As every basis element is an open set, of course the implication holds.

The second implication is slightly trickier. Suppose B intersects A for every basis element containing x. Well, we know any open set is made of unions of basis elements, and by consequence it means that any open set containing x must have one of such basis elements as a subset. As every basis elements with $x \in B$ intersects A, it follows that every open set containing x must intersect A. Thus, we know from the first item that $x \in \overline{A}$. ■

You might have noticed that we mentioned an "open set containing x" sometimes. In fact, this concept is quite common with Topology, and therefore it is handy for us to give a special name for such sets.

Definition 44 [Neighborhood]**:**

Let (X, τ) be a topological space. Let $x \in X$. We say a set $O \in \tau$ is a *neighborhood* of x if, and only if, $x \in O$. ♠

Theorem 43 [Another Possible Statement]**:**

Let (X, τ) be a topological space and let $A \subseteq X$. Then it holds that

i. $x \in \overline{A}$ *if, and only if, every neighborhood of* x *intersects* A*;*

ii. *if* $\mathfrak{B}$ *is a basis for the topology* τ*, then* $x \in \overline{A}$ *if, and only if,* B *intersects* $A, \forall B \in \mathfrak{B}; x \in \mathfrak{B}$. □

The second item still depends on some cumbersome notation, but the first one is certainly cleaner.

While we are here, we might as well define the concept of a neighborhood base for future reference.

Definition 45 [Neighborhood Basis]**:**

Let (X, τ) be a topological space. Let $x \in X$. A *neighborhood basis* for τ at x is a collection $\mathfrak{N} \subseteq \tau$ satisfying the following conditions:

i. $x \in \mathcal{B}, \forall \mathcal{B} \in \mathfrak{N}$;

ii. $O \in \tau, x \in O \Rightarrow \exists \mathcal{B} \in \mathfrak{N}; \mathcal{B} \subseteq O$. ♠

Examples [Calculating Closures]**:**

Consider the real line with the standard topology. The closure of any open interval I is the closed interval with the same extremes.

Let us write $I = (a, b)$. Of course $(a, b) \subseteq \overline{I}$, for $A \subseteq \overline{A}, \forall A \in \mathbb{P}(\mathbb{R})$.

We know the open intervals are a basis for the standard topology on $\mathbb{R}$. Thus, due to Theorem 43, we know that $x \in \overline{I}$ if, and only if, every open interval containing x intersects I.

Let $a \in (c, d)$. Then, by definition, $c < a < d$, and thus either $d \in I$ or $c < \frac{a+b}{2} < d$ and thus $\frac{a+b}{2} \in (c, d) \cap (a, b)$. A similar argument holds for b.

Suppose now that any other number is contained in $\overline{I}$. Let us call it x_0 and suppose, without any loss of generality, that $x_0 > b$. Then we pick the open interval $\left(\frac{x_0+b}{2}, x_0 + 1\right)$, for example. As $\frac{x_0+b}{2} > b$, every element of such interval is outside of I, and it follows from Theorem 43 that $x_0 \notin \bar{I}$.

Still on the same topological space, let us consider a larger set. Namely, $\mathbb{Q}$. What is $\overline{\mathbb{Q}}$?

Let $x \in \mathbb{R}$. We know $\mathbb{R}$ admits as a basis the set of open intervals centered at rational numbers with radius of the form $\frac{1}{n}, n \in \mathbb{N}^*$. As this is a basis, certainly there are intervals of this form that contain x (for $\mathbb{R}$ can be written as an arbitrary union of such intervals). However, by construction, such intervals always contain a rational number (namely, the center). Thus, by Theorem 43, every real number is an element of $\overline{\mathbb{Q}}$. As $\overline{\mathbb{Q}} \subseteq \mathbb{R}$, we conclude $\overline{\mathbb{Q}} = \mathbb{R}$. ♥

You might have noticed that we could have simply said that $\mathbb{Q}$ is dense in $\mathbb{R}$ and thus every interval of real numbers contains a rational number. I avoided such nomenclature for a simple reason: we also have a definition of a dense set within topology.

Definition 46 [Dense Set]**:**

Let (X, τ) be a topological space. We say a set $A \subseteq X$ is *dense* whenever it holds that $\overline{A} = X$. ♠

As usual, this is merely an extension of the similar concept known from metric spaces.

4 Limits of Sequences

As previous experiences with Real Analysis and Metric Spaces might suggest, the study of limits depends heavily on the notion of a sequence (which, of course, will also receive a more general formulation in terms of nets). Naturally, we should start this section by defining what is a sequence.

Definition 47 [Sequence]**:**

Let X be a non-empty set. A function $x\colon \mathbb{N} \to X$ is commonly called a *sequence* in X. ♠

Notation:

Instead of writing $x(n)$ for the image of $n \in \mathbb{N}$ through a sequence x, it is usual to write simply x_n. It is also customary to write $(x_n)_{n \geqslant 0}$, $(x_n)_{n \geqslant 1}$, $(x_n)_{n \in \mathbb{N}}$, *et cetera* for the sequence, instead of x. Some other notations can also be found (for example, referring to the sequence itself, not a the image of a natural n through the sequence, as x_n). ♦

You might already know the definition for the limit of a sequence in a metric space, for it is indeed very similar to the notion of limit introduced at Section 1.

Definition 48 [Convergence of a Sequence in a Metric Space]**:**

Let (M, d) be a metric space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in M. We say the sequence *converges* to a point $x \in M$ whenever it holds that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}; n > N \Rightarrow d(x_n, x) < \epsilon. \tag{4.1}$$

Notice that such $N \in \mathbb{N}$ may, and in general will, depend on ϵ. ♠

We want to make this definition more general and drop the dependence on a metric. After all, no notion of distance in available in topological spaces and we must work solely with notions about sets and open sets.

As the distance between the elements of the sequence decrease, we know that, in terms of sets, the terms of the sequence are contained in smaller and smaller sets. After all, the open balls are nested within each other. This suggests a definition of convergence based on the notion of neighborhoods: for every neighborhood O of the limit value (which we previously denoted as x) there should exist a natural number N such that $x_n \in O, \forall\, n > N$.

Nevertheless, not every sequence is convergent, but we could as well be interested in some other cases. For an example, the sequence $x_n = (-1)^n$ in the real line does not converge to any point, but it admits subsequences* that converge to either $+1$ or -1. Therefore, it is interesting for us to also develop the theory in the direction of understanding properties of subsequences of a sequence, even if the sequence itself does not converge to any value whatsoever.

Such ideas naturally bring us towards the definitions of points that are frequently and eventually at a set, which do not depend on the topological structure of the space.

Definition 49 [Frequently and Eventually]**:**

Let X be a non-empty set, $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $A \subseteq X$. We say $(x_n)_{n\in\mathbb{N}}$ is *frequently* in A if, and only if, there is an infinite amount of indices m such that $x_m \in A$. We say $(x_n)_{n\in\mathbb{N}}$ is *eventually* in A if, and only if, there is a natural number N such that $x_m \in A, \forall\, m > N$. ♠

Remark:

Notice that if a sequence is eventually in A, then it is frequently in A. However, the inverse does not hold: $x_n = (-1)^n$ is frequently in $(0, 2)$, but is is not eventually in such set. ♣

Well, I've said before that topology is all about limits and continuity, and therefore it would be weird if we did not need a topological structure to define limits. If we can talk about points that are eventually in a set or frequently in a set without any need for a topological structure, why did we prove so many results on open sets, bases, closures, and so on?

Suppose we were going to despise any topological structure and try to go for a definition based solely on set theory, no topology allowed. We would lose any and every notion of how close a point is to another. We discussed before an idea of trying to define a limit point by demanding that for every neighborhood there would be a "cutoff point" at our sequence such that every point from there onward would be inside that neighborhood. Well, now that any set is valid, even simple sequences lose their convergence properties. For example, is the sequence $\frac{1}{n}$ in the real line still convergent? If so, does it go to zero? In fact, without any notion of topology, $\{0, 1\}$ would become a neighborhood of 0 (we do not care about the set being open anymore, we don't even know what that means!). But so is $\{0, 2\}$, and the requirements for the sequence to be convergent would require that

*If $(x_n)_{n\in\mathbb{N}}$ is a sequence and $m \mapsto n_m$ is a crescent function from $\mathbb{N}$ to $\mathbb{N}$, $(x_{n_m})_{m\in\mathbb{N}}$ is said to be a *subsequence* of $(x_n)_{n\in\mathbb{N}}$.

the sequence is eventually constant, with value 0. This does not happen, and thus the sequence does not converge. Therefore, such a theory is simply not interesting at all, for is does not make our results more general. It restricts them, in fact. Hence, topology.

Therefore, we are motivated to define the concepts of cluster and limit points.

Definition 50 [Cluster Point]**:**

Let (X, τ) be a topological space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. A point $x \in X$ is said to be a *cluster point* of the sequence $(x_n)_{n\in\mathbb{N}}$ with respect to the topology τ if, and only if, $(x_n)_{n\in\mathbb{N}}$ is frequently in every neighborhood of x. ♠

Definition 51 [Limit Point]**:**

Let (X, τ) be a topological space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. A point $x \in X$ is said to be a *limit point* (sometimes called simply *limit*) of the sequence $(x_n)_{n\in\mathbb{N}}$ with respect to the topology τ if, and only if, $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood of x. ♠

Remark:

As being eventually in a set implies being frequently in the same set, it holds that, giving a sequence, every limit point of such sequence is a cluster point of the sequence. ♣

Example [Not Every Cluster Point is a Limit]**:**

There is no reason for a cluster point to be a limit point, but sometimes we tend to believe that such a property would be "likely", and pretend to do Mathematics on a probabilistic manner. Needless to say, this is likely to fail, but it is quite amusing to prove our intuition wrong through some counterexamples. After all, their existence justifies all the effort we have been putting into a theory. Thus, we shall prove that for quite a large family of sequences we can have the whole real line as the set of cluster points, albeit no point at all is a limit point.

Let $q \colon \mathbb{N} \to \mathbb{R}$ be a sequence such that $\operatorname{Ran} q = \mathbb{Q}$. Such a sequence does exist, for $\mathbb{Q}$ is a countable set. I claim that every real number is a cluster point of q with respect to the standard topology, but no real number is a limit point of q.

Let $x \in \mathbb{R}$. Let O_x be a neighborhood of x. As the open intervals with rational extremes are a basis for the standard topology in $\mathbb{R}$, we may simply prove that every such interval containing x has infinitely many rational numbers (and thus there are infinitely many terms of q in that interval).

Let $a, b \in \mathbb{Q}$ and such that $x \in (a, b)$. Let $m = \min\{x - a, b - x\}$. We know, from the Archimedean property of the real numbers, that there is a natural number n_0 such that $\frac{1}{n_0} < m$, and thus $x \in \left(a + \frac{1}{n_0}, b - \frac{1}{n_0}\right)$. It follows that $x \in \left(a + \frac{1}{n}, b - \frac{1}{n}\right), \forall\, n \in \mathbb{N}^*; n > n_0$. As $a + \frac{1}{n} \in \mathbb{Q}, \forall\, n \in \mathbb{N}^*$, we have found infinitely many rational numbers (*id est*, terms of q) in an arbitrary neighborhood of x. As the argument holds for every $x \in \mathbb{R}$, we have proven that every real number is a cluster point of $\mathbb{R}$.

We now want to prove that no real number is a limit point of q. In order to do so, suppose $x \in \mathbb{R}$ is a limit of q. Let me write m for the largest integer smaller than or equal to x. As x is a limit of q, it holds that there is a natural number $n_0 \in \mathbb{N}$ such that $q_n \in (m, m+1), \forall\, n \in \mathbb{N}; n > n_0$. However, this implies that there are no more than n_0

terms of q outside of $(m, m+1)$. Given that we have already proved that $(m+2, m+3)$, for example, has infinitely many terms of q, we have reached a contradiction and it is impossible for x to be a limit point of q. As the argument holds for every point $x \in \mathbb{R}$, q has no limit points in $\mathbb{R}$. ♥

Example [Not Every Sequence Has a Single Limit]**:**

As a second example, let us show that there are sequences in topological spaces that admit more than one limit (as opposed to what happens in metric spaces, when limits are always unique). In this example, we are going to consider the line with two origins: we add another element to the real line and introduce a topology in this space.

If we want to add another element to $\mathbb{R}$, we must pick some set which we already know to exist. As any set will do, let ✠ denote the Poor Fellow-Soldiers of Christ and of the Temple of Solomon (yes, the Templars). We write $X = \mathbb{R} \cup \{✠\}$.

As we are interested in topological properties, we still need to define a topology in X. Let $\mathfrak{B}_{\mathbb{R}}$ be the basis of open intervals for the standard topology in $\mathbb{R}$. Let $\mathfrak{B}_{✠} \equiv \{\{✠\} \cup B \backslash \{0\}\,; B \in \mathfrak{B}_{\mathbb{R}}\}$. We define $\mathfrak{B} \equiv \mathfrak{B}_{\mathbb{R}} \cup \mathfrak{B}_{✠}$. I claim $\mathfrak{B}$ is a basis for a topology in X. The details of the proof shall be left as an exercise, but I present a sketch.

By construction, $\mathfrak{B}$ covers X. In order to prove that $\mathfrak{B}$ is a basis, we still need to prove that $\forall\, B_1, B_2 \in \mathfrak{B}, \forall\, x \in B_1 \cap B_2, \exists\, B_3 \in \mathfrak{B}; x \in B_3 \subseteq B_1 \cap B_2$.

If we have either $B_1, B_2 \in \mathfrak{B}_{\mathbb{R}}$ or $B_1, B_2 \in \mathfrak{B}_{✠}$, the proof is trivial. We must then consider only the case in which each set is in a different collection. Let us suppose, without any loss of generality, that $B_1 = (a, b) \in \mathfrak{B}_{\mathbb{R}} \backslash \mathfrak{B}_{✠}$ and $B_2 = (c, d) \in \mathfrak{B}_{✠} \backslash \mathfrak{B}_{\mathbb{R}}$. We can simply split the intersection of both sets in negative and positive sides and reduce the problem to elements in $\mathfrak{B}_{\mathbb{R}}$.

Let us consider the sequence of elements of X given by $x_n = \frac{1}{n}, \forall\, n \in \mathbb{N}^*$. This sequence admits two limit points: 0 and ✠.

Indeed, let y denote either 0 or ✠. Consider an arbitrary neighborhood O of y such that $O \in \mathfrak{B}$, for simplicity. As any open set can be written as an union of elements of $\mathfrak{B}$, if we proof that x_n is eventually within any such neighborhood O, the argument holds for arbitrary neighborhoods and will follow that y is a limit point of x_n.

As $O \in \mathfrak{B}$, we can write it as $O = (a, b)_y$, denoting the interval starting at a, ending at b and containing only the origin y, but not the other one. For example, if $y = 0$, $0 \in (a, b)_y$, $✠ \notin (a, b)_y$, with an analogous relation for $y = ✠$.

Due to the Archimedean property of the real numbers, we know that, $\forall\, b > 0, \exists\, n_0 > \frac{1}{b}$, and therefore $b > \frac{1}{n_0}$. Thus, $\forall\, n > n_0, x_n < b$ and it follows that $x_n \in (a, b)_y, \forall\, n > n_0$. Hence, x_n is eventually in O, for any neighborhood O of y satisfying $O \in \mathfrak{B}$. As every neighborhood of y can be written as unions of such sets, it follows that x_n is eventually in any neighborhood of y, *id est*, y is a limit of x_n. As y is either 0 or ✠ and the argument holds for both, we conclude x_n admits two limits: 0 and ✠. ♥

The line with two origins presents a result which certainly seems odd, considering the usual properties limits respect in metric spaces. Namely, limits in topological spaces need not to be unique, albeit limits in metric spaces are always unique. There are topological spaces whose structure does not allow us to separate some points from others, and sequences in such spaces might admit more than one limit for a simple reason: both points are indistinguishable, from the topological point of view.

If we are interested in studying limits, this is actually quite a bummer. Eventually we could be interested in taking derivatives of functions (we are going to need more complicated spaces in order to do so) and it would be uninteresting for us to have a function with two derivatives at the same point, for we would want to give a general definition for the inclination of a function somewhere and to the procedure of finding tangent lines and planes and *et cetera*. Thus, if we have any dreams of making Calculus more general, we must first be sure the limits we taking are unique.

Motivated by the notion of trying to separate points, we are going to define the *Hausdorff property*.

Definition 52 [Hausdorff Spaces or T_2-Spaces]**:**

Let (X, τ) be a topological space. We say (X, τ) is a *Hausdorff space*, or a *T_2-Space*, or simply that (X, τ) satisfies the *Hausdorff property*, if, and only if, it holds that given two arbitrary points $x, y \in X$ there are disjoint open sets $O_x, O_y \in \tau$ such that O_x is a neighborhood of x and O_y is a neighborhood of y. ♠

Remark:

The name *T_2-Space* might seem a bit odd. If we consider it alone, it is indeed, but in fact the Hausdorff property is just one of the so called *Separation Axioms* (and yes, there is a definition for a T_0-Space, a T_1-Space, a $T_{3\frac{1}{2}}$-Space and actually quite a lot of options). Different axioms denote different separation properties which a space might or not obey. As an example, a T_0-Space satisfies the property that, given two points $x, y \in X$, at least one of them admits a neighborhood that does not contain the other. For now, we are interested exclusively in the Hausdorff property (which is often the most interesting), but more details concerning other separation axioms are available at Section 6. ♣

Naturally, the next step we should give is proving that such a property does solve the problem we had.

Theorem 53:

Let (X, τ) be a Hausdorff space. Then every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X admits at most one limit point. Furthermore, if it exists, such limit point is the only cluster point of $(x_n)_{n \in \mathbb{N}}$. □

Proof:

Suppose there is $x \in X$ such that x is a limit of $(x_n)_{n \in \mathbb{N}}$. Let $y \in X$ such that $x \neq y$. As (X, τ) is a Hausdorff space, there are disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. Given that x is a limit of $(x_n)_{n \in \mathbb{N}}$, we know that $\exists\, n_0 \in \mathbb{N}; x_n \in O_x, \forall\, n > n_0$. As O_x and O_y are disjoint, it follows that $x_n \notin O_y, \forall\, n > n_0$, and thus there are no more than n_0 terms of $(x_n)_{n \in \mathbb{N}}$. Therefore, we have found a neighborhood of y with a finite numbers of terms of $(x_n)_{n \in \mathbb{N}}$, which guarantees that y is not a cluster point of $(x_n)_{n \in \mathbb{N}}$. As every limit point is a cluster point and y is not a cluster point of $(x_n)_{n \in \mathbb{N}}$, it follows by contrapositive that y is not a limit of $(x_n)_{n \in \mathbb{N}}$. As the argument holds for every point $y \in X, y \neq x$, we conclude that x is the only limit point of the sequence.

If $(x_n)_{n \in \mathbb{N}}$ admits no limits points, the result holds trivially. ■

Now that points are being distinguished from a topological point of view (since they can be separated by open sets), limits of sequences are finally unique.

Notice that the Hausdorff axiom is not essential for us to study limits, continuity or Topology in general. However, it is useful, for endowing spaces with this extra property allow us to obtain more interesting results. Generality is interesting from the point of view that having few assumptions allows us to apply our results to many different spaces, but it comes with the price of having less results.

5 What is Continuity?

Theorem 7 allows us to extend the definition of what is a continuous function through the following definition:

Definition 54 [Continuous Function]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. We say that f is a *continuous function* if, and only if, $f^{-1}(A) \in \tau_X, \forall\, A \in \tau_Y$. ♠

Despite Theorem 7, this definition might still be a bit nebulous. Thus, it could be useful for us to verify that this definition recovers what we expect from real functions.

Example [Topological Continuity]**:**

Consider the real line $\mathbb{R}$ with its usual metric topology (the topology of open balls for the metric $d(x, y) = |x - y|$). Let $\theta\colon \mathbb{R} \to \mathbb{R}$ denote Heaviside's step function:

$$\theta(x) = \begin{cases} 1, \text{ if } x > 0, \\ \frac{1}{2}, \text{ if } x = 0, \\ 0, \text{ if } x < 0. \end{cases} \tag{5.1}$$

We know that θ should be continuous at any point $x \neq 0$ and discontinuous at $x = 0$. Thus, if we consider its restrictions to $\mathbb{R}^*_+ \equiv \{x > 0; x \in \mathbb{R}\}$ or $\mathbb{R}^*_- \equiv \{x < 0; x \in \mathbb{R}\}$, it should be a continuous function.

Let us pick an open set in the range of θ such that its preimage is either in $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$. For example, let us pick $A = \mathcal{B}_{\frac{1}{2}}(1)$, which is an open ball in $\mathbb{R}$ and thus is an open set in $\mathbb{R}$ (Lemma 5). $\theta^{-1}(A) = \mathbb{R}^*_+$. Since $\mathbb{R}^*_+ = \bigcup_{n \in \mathbb{N}^*} \mathcal{B}_1(n)$, where $\mathbb{N}^*$ denotes the positive natural numbers, $\mathbb{R}^*_+$ is indeed open and we see no problem on this region*.

Let us now pick an open set that might give us a bit more trouble, for example one that has 0 in its preimage. We might choose $A = \mathcal{B}_{\frac{1}{2}}\left(\frac{1}{2}\right)$, for example. Now we have that $\theta^{-1}(A) = \{0\}$, which is not an open set (since we are dealing with a metric topology, an easy way to see it is by proving that no open ball centered at 0 (which is the only element in the set) can be contained in $\{0\}$). Thus, θ can't be continuous on $\mathbb{R}$.

Notice now how the continuity of a function depends on the topology we are considering: if we had chosen the discrete topology instead of the usual topology, θ (and in fact any other function) would be a continuous function. ♥

*Be careful: this argument alone doesn't imply that θ is continuous on $\mathbb{R}^*_+$! Recall that continuity requires for $\theta^{-1}(A)$ to be open for *any* open set A, not only the one we've picked.

One might also wonder whether we could write the definition as "$f(A)$ is open for every open set $A \in X$" for a function $f\colon X \to Y$. The answer is no, and we might give an example of when this fails:

Example:

Once more let us pick $\mathbb{R}$ with its usual topology. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2, \forall\, x \in \mathbb{R}$, which is continuous everywhere when we consider the definition of continuity commonly used in Real Analysis. We pick the open set $A = \mathcal{B}_\epsilon(0)$ for any $\epsilon > 0$. Notice that $f(A) = [0, \epsilon)$, which is not an open set ♥

We also have a name for functions ϕ such that $\phi(A)$ is open for every open set A:

Definition 55 [Open Maps]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. We say that f is a *open* if, and only if, $f(A) \in \tau_Y, \forall\, A \in \tau_X$. ♠

Proposition 56:

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $Z \subseteq X$. If $f\colon X \to Y$ is a continuous function, its restriction $f\big|_Z : Z \to Y$ is continuous when Z is equipped with the relative topology. □

Proof:

We shall denote $g \equiv f\big|_Z$.

Let $O \in \tau_Y$. We want to prove that $g^{-1}(O)$ is open in the relative topology of Z. Since f is continuous, we know that $f^{-1}(O) \in \tau_X$.

Notice that

$$\begin{aligned} f^{-1}(O) &= \{x \in X; f(x) \in O\}, \\ g^{-1}(O) &= \{x \in Z \subseteq X; f(x) \in O\}. \end{aligned} \tag{5.2}$$

Thus, $g^{-1}(O) = f^{-1}(O) \cap Z$. Since $f^{-1}(O)$ is open relatively to X, the definition of the relative topology implies $g^{-1}(O)$ is open in Z, concluding the proof. ■

Of course, it is not unusual within Real Analysis for one to talk about a function continuous *at a given point*. Naturally, we may wonder how can we define continuity at a given point. The trick is simple: continuity at x used to be defined as $\forall\, \epsilon > 0, \exists\, \delta > 0; d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$. We must simply erase the open balls and write neighborhoods instead.

Definition 57 [Continuity at a Point]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. Let $x \in X$. We say f is *continuous at* x if, and only if, for every neighborhood O of $f(x)$ there is a neighborhood U of x such that $f(U) \subseteq O$. ♠

Proposition 58:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. Let $x \in X$. If for every neighborhood O of $f(x)$, $f^{-1}(O)$ is a neighborhood of x, then f is continuous at x. □

Proof:

Suppose that for every neighborhood O of $f(x)$ it holds that $f^{-1}(O)$ is a neighborhood of x. Then notice that $f\big(f^{-1}(O)\big) = O \subseteq O$. Thus, for every neighborhood O of $f(x)$ there is a neighborhood U of x such that $f(U) \subseteq O$. ■

A question that arises now is whether this new definition is compatible with the definition of continuity provided for topological spaces as a whole. This is settled in the following result:

Proposition 59:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. f is continuous if, and only if, f is continuous at x, $\forall\, x \in X$. □

Proof:

$\Rightarrow$: Let $x \in X$. Provided that f is continuous, it holds that $f^{-1}(O) \in \tau_Y, \forall\, O \in \tau_X$. Therefore, if O is a neighborhood of $f(x)$, then $f^{-1}(O)$ is an open set. Since $f(x) \in O$, it also holds that $x \in f^{-1}(O)$. Thus, $f^{-1}(O)$ is a neighborhood of x and it holds that f is continuous at x.

$\Leftarrow$: Let $O \in \tau_Y$. If $f^{-1}(O) = \varnothing$, then $f^{-1}(O)$ is trivially an open set and the proof is complete. Otherwise, let $xinf^{-1}(O)$. Notice that O is a neighborhood of $f(x)$. Since f is continuous at every point of X, it is continuous at x and it holds that $f^{-1}(O)$ is a neighborhood of x. Therefore, $f^{-1}(O)$ is an open set and the proof is complete. ■

Of course, we could as well state the definition of continuity in terms of the closure operator or in terms of closed sets, instead of open sets.

Theorem 60:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a function. The following statements are equivalent:

i. *for every open set O in Y, $f^{-1}(O)$ is a open set in X;*

ii. *for every closed set F in Y, $f^{-1}(F)$ is a closed set in X;*

iii. $\forall\, A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. □

Proof:

Let us first prove that if the first statement holds, then so does the second. Let F be a closed set in Y. Then $F^c \in \tau_Y$. Since the preimage of every open set under f is an open set, we know that $f^{-1}(F^c) = f^{-1}(F)^c \in \tau_X$. Therefore, $f^{-1}(F)$ is a closed set in X.

Assuming the second statement, we might as well prove the first in a similar fashion. Let O be a closed set in Y. Then O^c is a closed set. Since the preimage of every closed set under f is a closed set, we know that $f^{-1}(O^c) = f^{-1}(O)^c$ is a closed set. Therefore, $f^{-1}(O)$ is an open set in X.

Assuming the second statement, we now want to prove the third. Let $A \subseteq X$. Then we have

$$
\begin{aligned}
f(A) &\subseteq \overline{f(A)}, \\
A \subseteq f^{-1}(f(A)) &\subseteq f^{-1}\left(\overline{f(A)}\right), \\
A &\subseteq f^{-1}\left(\overline{f(A)}\right), \\
\overline{A} &\subseteq \overline{f^{-1}\left(\overline{f(A)}\right)}.
\end{aligned} \tag{5.3}
$$

Since the preimage of a closed set under f is a closed set, we know that $f^{-1}\left(\overline{f(A)}\right)$ is a closed set. As a consequence, $f^{-1}\left(\overline{f(A)}\right) = \overline{f^{-1}\left(\overline{f(A)}\right)}$. It follows that

$$
\begin{aligned}
\overline{A} &\subseteq f^{-1}\left(\overline{f(A)}\right), \\
f\left(\overline{A}\right) &\subseteq f\left(f^{-1}\left(\overline{f(A)}\right)\right), \\
f\left(\overline{A}\right) &\subseteq \overline{f(A)},
\end{aligned} \tag{5.4}
$$

as desired.

Finally, we want to prove that the third statement implies the second. Let F be a closed set in Y. We know that $\forall\, A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$, and therefore we have

$$
\begin{aligned}
f\left(\overline{f^{-1}(F)}\right) &\subseteq \overline{f\left(f^{-1}(F)\right)} = \overline{F} = F, \\
f\left(\overline{f^{-1}(F)}\right) &\subseteq F, \\
\overline{f^{-1}(F)} \subseteq f^{-1}\left(f\left(\overline{f^{-1}(F)}\right)\right) &\subseteq f^{-1}(F), \\
\overline{f^{-1}(F)} &\subseteq f^{-1}(F).
\end{aligned} \tag{5.5}
$$

Since $A \subseteq \overline{A}, \forall\, A \subseteq X$, it follows that $\overline{f^{-1}(F)} = f^{-1}(F)$ and, therefore, $f^{-1}(F)$ is a closed set in X. ■

Just as is the case for real functions, the composition of continuous functions is a continuous function itself.

Proposition 61:

Let (X, τ_X), (Y, τ_Y) *and* (Z, τ_Z) *be topological spaces and let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be continuous functions. Then the function* $g \circ f\colon X \to Z$ *is also a continuous function.* □

Proof:

Let $O \in \tau_Z$. Since g is continuous, $g^{-1}(O) \in \tau_Y$. Since f is continuous, $f^{-1}\left(g^{-1}(O)\right) \in \tau_X$. Since $f^{-1}\left(g^{-1}(O)\right) = (g \circ f)^{-1}(O)$, we conclude the function $g \circ f$ is continuous. ■

If you recall, continuity was the main reason we started studying Topology in the first place. It would not be surprising if the notion of a continuous function could now lead us into deeper insights when considering the relations between different topological spaces, just like isomorphisms allow us to find "hidden" relations between linear spaces.

Indeed, continuity allows us to define what is a homeomorphism, which is the appropriate equivalence relation when we are dealing with topological spaces.

Definition 62 [Homeomorphism]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a bijective function. Suppose further that both f and f^{-1} are continuous functions. Under these conditions, we say that f is a *homeomorphism* and the topological spaces (X, τ_X) and (Y, τ_Y) are said to be *homeomorphic*. ♠

Remark:

Perhaps you would expect that is a function is continuous, then so should its inverse be. However, let $O \in \tau_X$. The preimage of O under $f^{-1}\colon Y \to X$ is the set $\{y \in Y; f^{-1}(y) \in O\}$, which is equal to $f(O) = \{f(x); x \in O\}$. Thus, f^{-1} being continuous means that if O is an open set, then so is its image under f. We already know that this does not follow from continuity, and therefore the requirement is not superfluous.

Combining this with the fact that f being continuous means that the preimage of an open set is also an open set, we see that we are demanding that $O \in \tau_X \Rightarrow f(O) \in \tau_Y$ and $O \in \tau_Y \Rightarrow f^{-1}(O) \in \tau_X$, *id est*, $O \in \tau_X \Leftrightarrow f(O) \in \tau_Y$. The homeomorphism provides us not only with a identification between points on the topological space, but also "translates" the topology in X to the topology in Y and vice-versa. This means that every property of X that can be entirely expressed in terms of the topology τ_X also must hold for Y due to the homeomorphism. Such properties are called *topological properties*. ♣

We might as well define a similar notion for the case in which a topological space is "larger" than the other.

Definition 63 [Embedding]**:**

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \to Y$ be a surjective function. Let $f'\colon X \to f(X)$ be a function such that $f(x) = f'(x), \forall\, x \in X$, where $f(X)$ is considered as a topological subspace of Y. If f' is a homeomorphism, we say that f is an *embedding of* X *in* Y. ♠

Theorem 64:

Let (X, τ_X) *and* (Y, τ_Y) *be topological spaces. The relation* $(X, \tau_X) \simeq (Y, \tau_Y) \Leftrightarrow (X, \tau_X)$ *and* (X, τ_X) *are homeomorphic is an equivalence relation.* □

Proof:

Consider first the identity mapping in the topological space (X, τ_X), given by $i_X\colon X \to X$ such that $i_X(x) = x$. Notice that this map is bijective and it has itself as its inverse. Furthermore, it is continuous, for if we are given an open set $O \in \tau_X$, $i_X^{-1}(O) = O \in \tau_X$. Therefore, this is a continuous map with continuous inverse, and therefore it is a homeomorphism, which means (X, τ_X) is homeomorphic to itself, *id est*, $(X, \tau_X) \simeq (X, \tau_X)$.

Let now (X, τ_X) and (Y, τ_Y) be homeomorphic topological spaces and let $f\colon X \to Y$ be a homeomorphism. f is invertible, so let us consider the function $f^{-1}\colon Y \to X$. As f is a homeomorphism, f^{-1} is continuous. f^{-1} is invertible and has f, which is continuous, as its inverse. Thus, f^{-1} is a continuous function with continuous inverse, and therefore it is a homeomorphism between (Y, τ_Y) and (X, τ_X). Thus, (Y, τ_Y) and (X, τ_X) are homeomorphic and we see that $(X, \tau_X) \simeq (Y, \tau_Y) \Rightarrow (Y, \tau_Y) \simeq (X, \tau_X)$.

Finally, assume (X, τ_X), (Y, τ_Y) and (Z, τ_Z) are topological spaces such that $(X, \tau_X) \simeq (Y, \tau_Y)$ and $(Y, \tau_Y) \simeq (Z, \tau_Z)$. Let us denote $f\colon X \to Y$ for the homeomorphism between (X, τ_X) and (Y, τ_Y) and $g\colon Y \to Z$ for the homeomorphism between (Y, τ_Y) and (Z, τ_Z). We want to prove that (X, τ_X) and (Z, τ_Z) are homeomorphic.

Consider the map $g \circ f\colon X \to Z$. We know from Proposition 61 that $g \circ f$ is a continuous function. Since both f and g are bijective, so is $g \circ f$, which has $f^{-1} \circ g^{-1}$ as its inverse. Since f^{-1} and g^{-1} are continuous, so is $f^{-1} \circ g^{-1}$. Thus, $g \circ f$ is a continuous function with continuous inverse, which means (X, τ_X) and (Z, τ_Z) are homeomorphic, *id est*, $(X, \tau_X) \simeq (Z, \tau_Z)$.

We see then that the following properties hold for $\simeq$ for every topological spaces (X, τ_X), (Y, τ_Y) and (Z, τ_Z):

i. $(X, \tau_X) \simeq (X, \tau_X)$;

ii. $(X, \tau_X) \simeq (Y, \tau_Y) \Rightarrow (Y, \tau_Y) \simeq (X, \tau_X)$;

iii. $(X, \tau_X) \simeq (Y, \tau_Y)$ and $(Y, \tau_Y) \simeq (Z, \tau_Z) \Rightarrow (X, \tau_X) \simeq (Z, \tau_Z)$.

Therefore, $\simeq$ is indeed an equivalence relation. ■

Proposition 65:

Let $a, b \in \mathbb{R}$, $a < b$. The real line $\mathbb{R}$ equipped with the standard topology is homeomorphic to the interval (a, b) with the relative topology. □

Proof:

Consider the function $f\colon (a, b) \to \mathbb{R}$ given by

$$f(x) \equiv \tan\left(\frac{\pi}{b-a}x + \frac{\pi}{2}\left(\frac{a+b}{a-b}\right)\right). \tag{5.6}$$

We know from Real Analysis that such a function is a continuous bijection between $\mathbb{R}$ and (a, b) with continuous inverse. Thus, the result is proven. ■

Continuous functions are also interesting because they provide us with yet another way of specifying the topology on a space.

Theorem 66:

Let Λ be an arbitrary collection of indices, X be a set and let $(Y_\lambda, \tau_\lambda)$ be topological spaces, $\forall\, \lambda \in \Lambda$. Let $\{f_\lambda\colon X \to Y_\lambda\}_{\lambda \in \Lambda}$ be a family of maps. Then there is a unique coarsest topology τ_X in X which makes every function f_λ continuous. □

Proof:

Consider the family of all topologies over X that make f_λ continuous, $\forall\, \lambda \in \Lambda$, denoted $\mathfrak{T}$. This is a set, for it is merely a restriction of the set $\mathbb{P}(\mathbb{P}(X))$. We might then just consider the intersection of all elements of $\mathfrak{T}$, denoted by $\tau_X = \bigcap_{\tau \in \mathfrak{T}} \tau$. We know the intersection of topologies is a topology, and therefore τ_X is a topology over X. Notice that τ_X is coarser than any element of $\mathfrak{T}$ by construction. We now only need to prove that τ_X does make f_λ continuous, $\forall\, \lambda \in \Lambda$.

Pick $\lambda \in \Lambda$. Let $O \in \tau_\lambda$. We know that $f_\lambda^{-1}(O) \in \tau, \forall \tau \in \mathfrak{T}$, for every topology in $\mathfrak{T}$ makes f_λ continuous. Thus, $O \in \bigcap_{\tau\in\mathfrak{T}} \tau = \tau_X$. Therefore, the preimage of any open set in Y_λ under f_λ is in τ_X, which means f_λ if continuous when we equip X with τ_X. Since the result holds for every $\lambda \in \Lambda$, the proof is complete. ■

Definition 67 [Weak Topology]**:**

Let Λ be an arbitrary collection of indices, X be a set and let $(Y_\lambda, \tau_\lambda)$ be topological spaces, $\forall \lambda \in \Lambda$. Let $\{f_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$ be a family of maps. The coarsest topology τ_X in X which makes every function f_λ continuous is said to be the *weak topology* generated by $\{f_\lambda\}_{\lambda\in\Lambda}$. ♠

Definition 68 [Product Topology]**:**

Let Λ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and let $X = \times_{\lambda\in\Lambda} X_\lambda$. The *product topology* in X is the weak topology generated by the projections $\{\pi_\lambda : X \to X_\lambda\}_{\lambda\in\Lambda}$. ♠

Remark:

We provide below the definitions of the generalized Cartesian product, $\times_{\lambda\in\Lambda} X_\lambda$, and of the projections π_λ in the same situation for completeness. ♣

Definition 69 [Generalized Cartesian Product]**:**

Let Λ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. We define the *Cartesian product* of $\{X_\lambda\}_{\lambda\in\Lambda}$, $\times_{\lambda\in\Lambda} X_\lambda$, as the set

$$\bigtimes_{\lambda\in\Lambda} X_\lambda = \left\{ f : \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda ; f(\lambda) \in X_\lambda, f \text{ is a function} \right\}.$$

Let us write $X \equiv \times_{\lambda\in\Lambda} X_\lambda$ for simplicity for the rest of this definition. We define the *projections* $\pi_\lambda : X \to X_\lambda$ as the functions such that $\pi_\lambda(f) = f(\lambda)$.

From now on, when writing a generalized Cartesian product explicitly, we will not write the requirement that f is a function, but it shall always be understood implicitly. ♠

Lemma 70:

Let Λ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_\lambda\}_{\lambda\in\Lambda}$ be two families of sets. It holds that

$$\bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda) = \left(\bigtimes_{\lambda\in\Lambda} X_\lambda \right) \cap \left(\bigtimes_{\lambda\in\Lambda} Y_\lambda \right).$$

□

Proof:

Notice that

$$\bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda) = \left\{ f : \Lambda \to \bigcup_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda) ; f(\lambda) \in (X_\lambda \cap Y_\lambda) \right\}.$$

Furthermore,

$$\left(\bigtimes_{\lambda\in\Lambda} X_\lambda\right) \cap \left(\bigtimes_{\lambda\in\Lambda} Y_\lambda\right) = \left\{f\colon \Lambda \to \left(\bigcup_{\lambda\in\Lambda} X_\lambda\right) \cap \left(\bigcup_{\lambda\in\Lambda} Y_\lambda\right); f(\lambda) \in (X_\lambda \cap Y_\lambda)\right\},$$
$$= \left\{f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda); f(\lambda) \in (X_\lambda \cap Y_\lambda)\right\},$$
$$= \bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda).$$

This proves the result. ■

Theorem 71:

Let Λ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and let $X = \bigtimes_{\lambda\in\Lambda} X_\lambda$. The set

$$\mathfrak{B} = \left\{\bigtimes_{\lambda\in\Lambda} O_\lambda; O_\lambda \in \tau_\lambda, \forall\,\lambda \in \Lambda \text{ and } \{\lambda; O_\lambda \neq X_\lambda\}_{\lambda\in\Lambda} \text{ is finite}\right\}$$

is a basis for the product topology on X. □

Proof:

Notice that $\mathfrak{S} \equiv \left\{\pi_\lambda^{-1}(O_\lambda); O_\lambda \in \tau_\lambda, \forall\,\lambda \in \Lambda\right\}$ is a subbasis for a topology in X. Indeed, since $O_\lambda \subseteq X_\lambda, \forall\,\lambda \in \Lambda$, $\pi_\lambda^{-1}(O_\lambda) \subseteq \pi_\lambda^{-1}(X_\lambda) = X$. Since $\pi_\lambda^{-1}(X_\lambda) \in \mathfrak{S}, \forall\,\lambda \in \Lambda$, we have that $X \in \mathfrak{S}$ and thus it is clear that $\mathfrak{S}$ must be a subbasis for a topology in X.

We shall denote $U_\lambda^\lambda \equiv O_\lambda$ and $U_\lambda^\kappa \equiv X_\lambda, \forall\,\kappa \in \Lambda, \kappa \neq \lambda$. Let $L = \{\lambda \in \Lambda; O_\lambda \neq X_\lambda\}$. We assume, as stated in the definition of $\mathfrak{B}$, that L is a finite set.

The basis induced by this subbasis has elements given by

$$\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda) = \bigcap_{\lambda\in L} \bigtimes_{\kappa\in\Lambda} U_\lambda^\kappa,$$
$$= \bigtimes_{\lambda\in\Lambda} O_\lambda. \tag{5.7}$$

Therefore, the basis induced by $\mathfrak{S}$ is $\mathfrak{B}$. Since $\pi_\lambda^{-1}(O_\lambda)$ must be an open set in the product topology (otherwise, the projection would not be continuous), we see the topology generated by $\mathfrak{B}$ must be the product topology. The elements of the topology generated by $\mathfrak{B}$ are given by

$$O = \bigcup_{\mu\in M} \bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda^\mu), \tag{5.8}$$

where M is an arbitrary set of indices, $O_\lambda^\mu \in \tau_\lambda, \forall\,\mu \in M, \forall\,\lambda \in \Lambda$. All these sets must be open in the product topology, for the finite intersection and the arbitrary union of open sets are open sets as well. The product topology also cannot be properly contained in the topology generated by $\mathfrak{B}$, for the product topology must contain $\mathfrak{S}$ and be closed under finite intersections and arbitrary unions, which means the elements of the topology generated by $\mathfrak{B}$ must be elements of the product topology. Thus, $\mathfrak{B}$ does generate the product topology. ■

Lemma 72:

Let Λ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. Suppose there is $\lambda \in \Lambda$ such that $X_\lambda = \varnothing$. Then $X = \bigtimes_{\lambda\in\Lambda} X_\lambda = \varnothing$. □

Proof:

$f \in X \Rightarrow f(\lambda) \in X_\lambda = \varnothing$. Since $\forall\, x, x \notin \varnothing$, X must be empty. ■

Theorem 73:

Let Λ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of Hausdorff topological spaces and let $X = \bigtimes_{\lambda\in\Lambda} X_\lambda$. Then it holds that (X, τ) is Hausdorff, where τ denotes the product topology. □

Proof:

We want to prove that given two points $f, g \in X$ such that $f \neq g$, there are two disjoint open sets O, U such that $f \in O, g \in U$.

Since $f \neq g$, there is at least one $\lambda \in \Lambda$ such that $f(\lambda) \neq g(\lambda)$. Let us write $f(\lambda) = x_\lambda, g(\lambda) = y_\lambda$. Since X_λ is a Hausdorff space, there are two sets O_λ and U_λ such that $x_\lambda \in O_\lambda$ and $y_\lambda \in U_\lambda$ with $O_\lambda \cap U_\lambda = \varnothing$. Let us define $O_\kappa = U_\kappa = X_\kappa, \forall\, \kappa \in \Lambda; \kappa \neq \lambda$. Notice that $f \in \bigtimes_{\kappa\in\Lambda} O_\kappa$ and $g \in \bigtimes_{\kappa\in\Lambda} U_\kappa$. Furthermore,

$$\begin{aligned}\left(\bigtimes_{\kappa\in\Lambda} O_\kappa\right) \cap \left(\bigtimes_{\kappa\in\Lambda} U_\kappa\right) &= \bigtimes_{\kappa\in\Lambda} (O_\kappa \cap U_\kappa), \\ &= \varnothing. \end{aligned} \tag{5.9}$$

The last line follows from Lemma 72. ■

Proposition 74:

Let Λ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and (X, τ_X) be the product space generated by such a family. Let (Y, τ_Y) be a topological space as well. Given a function $f\colon Y \to X$, it holds that f is continuous if, and only if, $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$. □

Proof:

- $\Rightarrow$: Suppose f is continuous. By definition of (X, τ_X), we know that π_λ is continuous $\forall\, \lambda \in \Lambda$. Since the composition of continuous functions is continuous, $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$.

- $\Leftarrow$: Suppose $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$. Then given an open set $O_\lambda \in \tau_\lambda$, $f^{-1}\left(\pi_\lambda^{-1}(O_\lambda)\right) \in \tau_Y$. Since the product topology makes every projection a continuous function, we know $\pi_\lambda^{-1}(O_\lambda) \in \tau_X$. We now want to prove that $f^{-1}(O) \in \tau_Y, \forall\, O \in \tau_X$.

  We know from the proof to Theorem 71 that an arbitrary open set in X can be written as $\bigtimes_{\lambda\in\Lambda} O_\lambda = \bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)$, where $O_\lambda \in \tau_\lambda, \forall\, \lambda \in \Lambda, O_\kappa \neq X_\kappa, \forall\, \kappa \in L$ and $L \subseteq \Lambda$ is finite. Since

$$f^{-1}\left(\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)\right) = \bigcap_{\lambda\in L} f^{-1}\left(\pi_\lambda^{-1}(O_\lambda)\right), \tag{5.10}$$

$\pi_\lambda^{-1}(O_\lambda) \in \tau_X, \forall\, \lambda \in \Lambda$ and the finite intersection of open sets is an open set, we may conclude that $f^{-1}\left(\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)\right) \in \tau_Y$. As we mentioned earlier, any element of τ_X can be written in the form $\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)$, and thus the proof is complete.

■

Remark:

Notice that if we consider the Cartesian product of a series of *equal* spaces (*id est*, $X_\lambda = X, \forall\, \lambda \in \Lambda$, for some space X), then $\bigtimes_{\lambda\in\Lambda} X_\lambda$ is the set of all functions from Λ to X:

$$\begin{aligned} \bigtimes_{\lambda\in\Lambda} X_\lambda &= \left\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda ; f(\lambda) \in X_\lambda \right\}, \\ &= \{ f\colon \Lambda \to X ; f(\lambda) \in X \}, \\ &= \{ f\colon \Lambda \to X \} . \end{aligned} \tag{5.11}$$

♣

Notation [X^Λ]:

Due to the previous remark and the common notation for a finite Cartesian product: $X^n \equiv \bigtimes_{1\leqslant 1\leqslant n} X$, we shall write the set of all functions from a set Λ to a set X as $X^\Lambda \equiv \bigtimes_{\lambda\in\Lambda} X$. ♦

Proposition 75:

Let (X, τ_X) *be a topological space and* Λ *be an arbitrary set. Consider the topological space* (X^Λ, τ)*, where* τ *is the product topology. Given a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* X^Λ *and* $f \in X^\Lambda$*, it holds that* $f_n \to f$ *in the product topology if, and only if,* $f_n \to f$ *pointwise,* id est, $f_n(\lambda) \to f(\lambda), \forall\, \lambda \in \Lambda$. □

Proof:

⇒: Let us assume $f_n \to f$ in the product topology, *id est*, f_n is eventually in every neighborhood of f. We want to prove that $f_n \to f$ pointwise.

Let $O_\lambda \in \tau)_X$ be such that $O = \bigtimes_{\lambda\in\Lambda} O_\lambda$ is a neighborhood of f (notice that every neighborhood of f can be written in such a way). There is a natural number, n_0, such that $f_n \in O, \forall\, n \geqslant n_0$. Notice thus that $f(\lambda), f_n(\lambda) \in O_\lambda, \forall\, n \geqslant n_0$. Thus, O_λ is a neighborhood of $f(\lambda)$ and the sequence $(f_n(\lambda))_{n\in\mathbb{N}}$ is eventually in such a neighborhood. Notice that in O is a neighborhood of f if, and only if, every O_λ is be a neighborhood of $f(\lambda)$, and thus it is proven that convergence in the product topology implies pointwise convergence.

⇐: Suppose $f_n(\lambda) \to f(\lambda), \forall\, \lambda \in \Lambda$. This means that, for every neighborhood O_λ of $f(\lambda)$, there is a natural number $n_0(\lambda)$ such that $f_n(\lambda) \in O_\lambda, \forall\, n \geqslant n_0(\lambda)$.

For every λ, let O_λ be a neighborhood of $f(\lambda)$ such that $O = \bigtimes_{\lambda\in\Lambda} O_\lambda$ is an open set in X^Λ. Then O is a neighborhood of f. Let $n_0 = \max_{\lambda\in\Lambda} n_0(\lambda)$. Notice that $f_n(\lambda) \in O_\lambda, \forall\, n \geqslant n_0, \forall\, \lambda \in \Lambda$. Thus, $f_n \in O, \forall\, n \geqslant n_0$. Since every neighborhood of f can be written in the previous form, this means that f_n is eventually in any neighborhood of f, and thus $f_n \to f$ in the product topology. ■

Naturally, many of the functions we consider interesting have the real line or the complex plane as its codomain. We shall then introduce some notation.

Notation [$\tau_{\mathbb{C}}$]:

When considered as a set, without algebraic properties, $\mathbb{C} = \mathbb{R}^2$. Thus, the standard topology on $\mathbb{C}$, which we shall denote by τ_C, is the product topology obtained when we consider $(\mathbb{C}, \tau_{\mathbb{C}})$ as the product space of $(\mathbb{R}, \tau_{\mathbb{R}})$ with itself. ♦

Notation [$B(X)$, $C(X)$ and $BC(X)$]:

Let $\mathbb{F}$ be either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. Let (X, τ) be a topological space. We denote by $B(X, \mathbb{F})$ the set of all bounded functions $f: X \to \mathbb{F}$. Similarly, we denote by $C(X, \mathbb{F})$ the set of all continuous functions $f: X \to \mathbb{F}$ when X if equipped with the τ topology (which should be clear by context) and $\mathbb{F}$ is equipped with its standard topology. We might also consider the set

$$BC(X, \mathbb{F}) \equiv B(X, \mathbb{F}) \cap C(X, \mathbb{F}). \tag{5.12}$$

When considering complex-valued functions, we might drop the $\mathbb{C}$ and write simply

$$B(X) \equiv B(X, \mathbb{C}), \quad C(X) \equiv C(X, \mathbb{C}), \quad BC(X) \equiv BC(X, \mathbb{C}).$$ ♦

Lemma 76:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $\mathfrak{B}$ be a basis for a topology in Y generating τ_Y. A function $f: X \to Y$ is continuous if, and only if, $f^{-1}(\mathcal{B}) \in \tau_X, \forall \mathcal{B} \in \mathfrak{B}$. □

Proof:

- $\Rightarrow$: Suppose f is continuous. Then $f^{-1}(O) \in \tau_X, \forall O \in \tau_Y$. Since $\mathfrak{B} \subseteq \tau_Y$, the statement holds.

- $\Leftarrow$: Let us assume $f^{-1}(\mathcal{B}) \in \tau_X, \forall \mathcal{B} \in \mathfrak{B}$. We know every open set $O \in \tau_Y$ can be written as $\bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda$, where Λ is some set of indices and $\mathcal{B}_\lambda \in \mathfrak{B}, \forall \lambda \in \Lambda$. Thus, we have that

$$\begin{aligned} f^{-1}(O) &= f^{-1}\left(\bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda\right) \\ &= \bigcup_{\lambda \in \Lambda} f^{-1}(\mathcal{B}_\lambda). \end{aligned} \tag{5.13}$$

  We already know that $f^{-1}(\mathcal{B}_\lambda) \in \tau_X, \forall \lambda \in \Lambda$. Thus, since arbitrary unions of open sets are open sets, the result holds. ■

Lemma 77:

$\mathfrak{B} = \{(a, b) \times (c, d); a, b, c, d \in \mathbb{R}\}$ *is a basis for the product topology in* $\mathbb{R}^2$. □

Proof:

Let us first prove that $\mathfrak{B}$ is indeed a basis for a topology in $\mathbb{R}^2$.Let $(x, y) \in \mathbb{R}^2$. Since the open intervals are a basis for a topology in $\mathbb{R}$, we know that there are real numbers a, b, c, d such that $a < x < y$ and $c < y < d$, and therefore $(x, y) \in (a, b) \times (c, d)$.

Let now $\mathcal{B}_1 = (a,b) \times (c,d), \mathcal{B}_2 = (e,f) \times (g,h)$. Notice that $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$. We want to prove that, $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$.

Let $\mathcal{B}_3 = \mathcal{B}_1 \cap \mathcal{B}_2 = \big[(a,b) \cap (e,f)\big] \times \big[(c,d) \cap (g,h)\big]$. Since the intersection of open intervals is an open interval, $\mathcal{B}_3 \in \mathfrak{B}$. Clearly it holds that $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Therefore, $\mathfrak{B}$ is indeed a basis for a topology in $\mathbb{R}^2$.

We know that $\mathfrak{B}' = \{O \times U; O, U \in \tau_{\mathbb{R}}\}$ is a basis for the product topology in $\mathbb{R}^2$, thanks to Theorem 71. Notice that $\mathfrak{B} \subseteq \mathfrak{B}'$. Therefore, $\forall\, x \in \mathbb{R}^2$ and $\forall\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$ we know that there is $\mathcal{B}' \in \mathfrak{B}'$, given by $\mathcal{B}' = \mathcal{B}$, such that $x \in \mathcal{B}' \subseteq \mathcal{B}$. Therefore, due to Proposition 18, the topology generated by $\mathfrak{B}'$ is finer than the topology generated by $\mathfrak{B}$.

However, let $x \in \mathbb{R}^2$. We shall write $x = (x_1, x_2)$. Let $\mathcal{B}' \in \mathfrak{B}'$ such that $x \in \mathcal{B}'$. Suppose $\mathcal{B}' = O \times U$, where $O, U \in \tau_{\mathbb{R}}$. Since the open intervals are a basis for the standard topology in $\mathbb{R}$, we know there are families of indices Λ and M and real numbers $a_\lambda, b_\lambda, c_\mu, d_\mu, \forall\, \lambda \in \Lambda, \forall\, \mu \in M$ such that $O = \bigcup_{\lambda \in \Lambda}(a_\lambda, b_\lambda)$ and $U = \bigcup_{\mu \in M}(c_\mu, d_\mu)$. Therefore, since $x \in O \times U$, we know there is some $\lambda \in \Lambda$ and some $\mu \in M$ such that $x_1 \in (a_\lambda, b_\lambda)$ and $x_2 \in (c_\mu, d_\mu)$. Thus, $x \in (a_\lambda, b_\lambda) \times (c_\mu, d_\mu) \equiv \mathcal{B} \in \mathfrak{B}$, with $\mathcal{B} \subseteq \mathcal{B}'$. Therefore, the topology generated by $\mathfrak{B}$ is finer than the topology generated by $\mathfrak{B}'$. Since the topology generated by $\mathfrak{B}'$ is also finer than the topology generated by $\mathfrak{B}$, we conclude both topologies are in fact the same, and thus $\mathfrak{B}$ is a basis for the product topology in $\mathbb{R}^2$. ■

Lemma 78:

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f\colon X \to Y$ be a constant function, id est, $\exists\, y_0 \in Y; f(x) = y_0, \forall\, x \in X$. *Then f is continuous.* □

Proof:

Let $O \in \tau_Y$. If $y_0 \in O$, then $f^{-1}(O) = X \in \tau_X$. If $y_0 \notin O$, then $f^{-1}(O) = \varnothing \in \tau_X$. ■

Proposition 79:

Let (X, τ) be a topological space. $C(X)$ and $BC(X)$ can be regarded as complex vector spaces when equipped with the usual addition and multiplication of complex-valued functions. □

Proof:

We must first prove that addition and multiplication of complex-valued (bounded) continuous functions are binary operations in $C(X)$ and $BC(X)$. It is known that the addition and multiplication of bounded functions is always a bounded function, and thus we shall only bother with the proof of continuity.

Let $f, g \in C(X)$. We want to prove that $f + g \in C(X)$. In order to do so, let us consider a different function: let $h\colon X \to \mathbb{C}^2$ be the function defined by $h(x) = (f(x), g(x))$. Since f and g are continuous, Proposition 74 guarantees h is continuous as well when we equip $\mathbb{C}^2$ with the product topology.

Consider now the map $+_{\mathbb{C}}\colon \mathbb{C}^2 \to \mathbb{C}$ such that $+_{\mathbb{C}}(x,y) = x + y, \forall\, x, y \in \mathbb{C}$, *id est*, $+_{\mathbb{C}}$ is ordinary addition in the complex plane. $+_{\mathbb{C}}$ is a continuous function. In order to prove this, notice that given $x, y, z, w \in \mathbb{R}$, we have $+_{\mathbb{C}}\big((x,y),(z,w)\big) = (x+z, y+w) \in \mathbb{C}$. If we prove the coordinate functions $+(x,z) = x + z$ and $+(y,w) = y + w$, which are simply real addition, are continuous, then $+_{\mathbb{C}}$ is continuous due to Proposition 74.

Due to Lemma 76 and the fact that the standard topology on $\mathbb{R}$ is generated by the open intervals $(a,b) \subseteq \mathbb{R}$, we must only prove that $+^{-1}\big((a,b)\big)$ is an open set for any pair of real numbers $a, b \in \mathbb{R}$.

Notice that $+^{-1}\big((a,b)\big) = \big\{(x,y) \in \mathbb{R}^2; a < x+y < b\big\}$. We want to prove this is an open set. Due to Theorem 15 Lemma 77, we must simply prove that $\forall\, x, y \in \mathbb{R}; a < x+y < b, \exists\, c, d, e, f \in \mathbb{R}$ such that $(x,y) \in (c,d) \times (e,f)$ and $a < z+w < b, \forall\,(z,w) \in (c,d) \times (e,f)$.

Let $c = \frac{a-y+x}{2}, d = \frac{b-y+x}{2}, e = \frac{a-x+y}{2}, f = \frac{b-x+y}{2}$. Notice that $x > c$, for $x - c = \frac{y+x-a}{2} > 0$, for $x+y > a$. Analogous arguments guarantee that $x \in (c,d)$ and $y \in (e,f)$.

Let $z \in (c,d)$ and $w \in (e,f)$. Then

$$\begin{aligned} z+w &> \frac{a-y+x}{2} + \frac{a-x+y}{2}, \\ &= a. \end{aligned} \tag{5.14}$$

Similarly,

$$\begin{aligned} z+w &< \frac{b-y+x}{2} + \frac{b-x+y}{2}, \\ &= b. \end{aligned} \tag{5.15}$$

Thus, $+^{-1}\big((a,b)\big)$ is an open set, which implies addition, both real and complex, is a continuous function. Since $+_{\mathbb{C}}$ is continuous, the composition $+_{\mathbb{C}} \circ h \colon X \to \mathbb{C}$ is also continuous. However, notice that $(+_{\mathbb{C}} \circ h)(x) = f(x) + g(x) = (f+g)(x)$. It is thus proven that the sum of continuous complex-valued functions is a continuous function.

The proof that complex multiplication is continuous can be done in a similar manner. The same argument used to prove that the sum of continuous functions is continuous can be applied to show that the product of continuous functions is continuous as well.

We now know that usual addition is a binary operation in $C(X)$ and $BC(X)$ and that $C(X)$ and $BC(X)$ are closed under scalar multiplication (for complex scalars are simply constant functions, which are continuous due to Lemma 78). Let us now prove that they obey the conditions necessary for $C(X)$ and $BC(X)$ to be vector spaces.

- A1 Complex addition is associative, and therefore so is the addition of complex-valued functions;
- A2 The constant function $0(x) = 0, \forall\, x \in X$ is continuous due to Lemma 78 and is clearly bounded. Furthermore, $f + 0 = 0 + f = f$ for every continuous function f;
- A3 For every continuous function f, there is a function $f^* = -f$ such that $f + f^* = f^* + f = 0$;
- A4 Complex addition is commutative, and thus so is the addition of complex-valued functions;
- M1 Complex multiplication is associative, and therefore it holds that, $\forall\, x \in X, \forall\, z, w \in \mathbb{C}, \forall\, f \in C(X), [zw]f(x) = z[wf(x)]$;
- M2 We know that $1 \in \mathbb{C}$ is such that $1 \cdot f = f$;
- D1 Since complex multiplication is distributive over complex addition, we know that $\forall\, x \in X, \forall\, z, w \in \mathbb{C}, \forall\, f \in C(X), (z+w)f(x) = zf(x) + wf(x)$;

D2 Since complex multiplication is distributive over complex addition, we know that $\forall\, x \in X, \forall\, z \in \mathbb{C}, \forall\, f, g \in C(X), z[f+g](x) = zf(x) + zg(x)$.

Since these properties hold in $C(X)$, they also hold in $BC(X)$. 0 is bounded and the operations we defined are closed in $BC(X)$, ensuring $BC(X)$ is a subspace of $C(X)$. This concludes the proof. ■

The proof to Proposition 79 also showed the following results:

Theorem 80:

Let $\mathbb{F}$ be either the real line or the complex plane and consider it equipped with the standard topology. The functions $+ \colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot \colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ defined as $+(x, y) = x + y$ and $\cdot(x, y) = x \cdot y$ are continuous when $\mathbb{F}^2$ is equipped with the product topology. □

Proof:

See proof to Proposition 79. ■

Definition 81 [Uniform Norm and Metric]**:**

Let (X, τ) be a topological space. Let $f \in B(X)$. We define the *uniform norm* of f, denoted $\|f\|_u$, through

$$\|f\|_u = \sup_{x \in X} |f(x)|. \tag{5.16}$$

We define the *uniform metric* on $B(X)$ through $d(f, g) = \|f - g\|_u$. ♠

Proposition 82:

Let (X, τ) be a topological space. The function $d \colon B(X) \times B(X) \to \mathbb{R}_+$ given by $d(f, g) = \|f - g\|_u$ defines a metric in $B(X)$. □

Proof:

Let $f, g, h \in B(X)$. Notice that

$$\begin{aligned} d(f, g) &= \|f - g\|_u, \\ &= \sup_{x \in X} |f(x) - g(x)|, \\ &= \sup_{x \in X} |g(x) - f(x)|, \\ &= \|g - f\|_u, \\ &= d(g, f). \end{aligned} \tag{5.17}$$

It is clear that $f = g$ implies $d(f, g) = 0$, for $f(x) - f(x) = 0$ and $\sup_{x \in X} 0 = 0$. Furthermore, notice that $\forall\, x \in X, 0 \leqslant |f(x) - g(x)| \leqslant \|f - g\|_u = d(f, g)$. Thus, if $d(f, g) = 0, 0 \leqslant |f(x) - g(x)| \leqslant 0$ and we see that it holds that $d(f, g) = 0 \Leftrightarrow f = g$.

Finally, we must prove the triangle inequality. Notice that, if we write $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$, we have that $|z| = \sqrt{x^2 + y^2}$. This means that $|z - w| = d_2(z, w)$, where d_2 stands for the Euclidean metric, which we know satisfies the triangle inequality.

Keeping this in mind, notice that:

$$\begin{aligned}
d(f,g) &= \sup_{x\in X} |f(x) - g(x)|, \\
&= \sup_{x\in X} \left[d_2\left(f(x), g(x)\right) \right], \\
&\leqslant \sup_{x\in X} \left[d_2\left(f(x), h(x)\right) + d_2\left(g(x), h(x)\right) \right], \\
&\leqslant \sup_{x\in X} \left[d_2\left(f(x), h(x)\right) \right] + \sup_{x\in X} \left[d_2\left(g(x), h(x)\right) \right], \\
&= d(f,h) + d(g,h).
\end{aligned} \tag{5.18}$$

Consequently, d defines a metric in $B(X)$. ■

Remark:

Notice that the hypothesis that $f \in B(X)$ in the definition of the uniform norm is necessary if we want it to be well-defined. If we picked, *exempli gratia*, $f \in C(X)$, the supremum could be ill-defined, for the set $\{|f(x)|; x \in X\}$ would possibly be unbounded. ♣

An interesting result concerning the uniform metric requires us to define some convergence criteria on metric spaces.

Definition 83 [Modes of Convergence on Metric Spaces]**:**

Let X be a set and (M, d) be a metric space. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \colon X \to M$. Let $f \colon X \to M$ be a function.

i. We say f_n converges *pointwise* to f whenever it holds that

$$\forall\, x \in X, \forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d\left(f_n(x) - f(x)\right) < \epsilon; \tag{5.19}$$

ii. We say f_n converges *uniformly* to f whenever it holds that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, d\left(f_n(x) - f(x)\right) < \epsilon. \tag{5.20}$$

Notice that uniform convergence implies pointwise convergence. ♠

Proposition 84:

Let (X, τ) *be a topological space. Convergence in* $B(X)$ *with respect to the uniform metric is equivalent to uniform convergence in* X, id est, *if* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of functions* $f_n \colon X \to \mathbb{C}$ *and* $f \colon X \to \mathbb{C}$ *is a function, then*

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon \tag{5.21}$$

if, and only if,

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \epsilon, \tag{5.22}$$

where d *denotes the uniform metric on* $B(X)$. □

Proof:

$\Rightarrow$: Assume $f_n \to f$ in the uniform metric. Let $\epsilon > 0$. Then we know that $\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon$, *id est*, $\forall\, n > n_0, \sup_{x \in X} |f_n(x) - f(x)| < \epsilon$. Since $\forall\, x \in X, |f_n(x) - f(x)| \leqslant \sup_{x \in X} |f_n(x) - f(x)|$, we see that is must hold that $\forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \epsilon$, proving $f_n \to f$ uniformly.

$\Leftarrow$: Suppose $f_n \to f$ uniformly. Let $\forall\, \epsilon > 0$. We know that $\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Since $\forall\, x \in X, |f_n(x) - f(x)| < \frac{\epsilon}{2}$, it holds that $\sup_{x \in X} |f_n(x) - f(x)| \leqslant \frac{\epsilon}{2} < \epsilon$. Thus, we see that $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon$ and we conclude that uniform convergence implies convergence in the uniform metric. ■

We might also explore it even further and see that $B(X)$ is in fact a *complete* space, which essentially means a space with no "holes". In order to understand this statement, we must work again with some concepts regarding metric spaces.

Definition 85 [Cauchy Sequences]**:**

Let (M, d) be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of M. $(x_n)_{n \in \mathbb{N}}$ is said to be a *Cauchy sequence* if, and only if, it holds that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, d(x_n, x_m) < \epsilon. \tag{5.23}$$

♠

Proposition 86:

Let (M, d) be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of M. If $(x_n)_{n \in \mathbb{N}}$ is convergent, it is a Cauchy sequence. □

Proof:

Let $x \in M$ be such that $x_n \to x$. We know that, given $\epsilon > 0$,

$$\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(x_n, x) < \frac{\epsilon}{2}. \tag{5.24}$$

Let $n, m > n_0$. Then we have that

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x) + d(x_m, x), \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned} \tag{5.25}$$

Therefore, we see that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, d(x_n, x_m) < \epsilon. \tag{5.26}$$

■

Example [Divergent Cauchy Sequence]**:**

Not every Cauchy sequence is convergent. In fact, spaces in which Cauchy implies convergence are said to be *complete*. As a simple example, consider the metric space $(\mathbb{Q}, d_2)$: the rational numbers with the Euclidean metric $d_2(x, y) = |x - y|$.

Let us define $x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$, where $\forall\, x \in \mathbb{R}, \lfloor x \rfloor$ denotes the largest integer smaller than x. Notice the first few elements of this sequence are $x_0 = 1, x_1 = 1.4, x_2 = 1.41, x_3 = 1.414, x_4 = 1.4142$, *et cetera*. The sequence consists of the decimal expansion of the square root of two up to the n-th decimal place. Naturally, $x_n \to \sqrt{2}$, which is not a rational number, despite x_n being a Cauchy sequence, a fact I leave for you to check. ♥

Definition 87 [Completeness of a Metric Space]:

Let (M, d) be a metric space. We say (M, d) is *complete* if, and only if, every Cauchy sequence of elements of M is convergent. ♠

Proposition 88:

Let (X, τ) *be a topological space.* $B(X)$ *is complete in the uniform metric.* □

Proof:

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of functions $f_n : X \to \mathbb{C}$, *id est*, $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, d(f_n, f_m) < \epsilon$. Since $d(f_n, f_m) = \sup_{x\in X} |f_n(x) - f_m(x)|$ and $\forall\, x \in X, |f_n(x) - f_m(x)| \leqslant \sup_{x\in X} |f_n(x) - f_m(x)|$, we see that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, \forall\, x \in X, |f_n(x) - f_m(x)| < \epsilon. \tag{5.27}$$

Therefore, if we consider a fixed point $x \in X$, the sequence of real numbers determined by $\left(f_n(x)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Every Cauchy sequence in the complex plane is convergent, and thus so is $\left(f_n(x)\right)_{n\in\mathbb{N}}$. Let us define $f(x) = \lim_{n\to\infty} f_n(x)$.

Let us now allow $m \to \infty$ in Eq. (5.27). It follows that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \epsilon. \tag{5.28}$$

Therefore,

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \sup_{x\in X} |f_n(x) - f(x)| < \epsilon, \tag{5.29}$$

which means $f_n \to f$ with respect to the metric d. ■

Lemma 89:

Every metric space (M, d) *satisfies the Hausdorff property.* □

Proof:

Let $x, y \in M, x \neq y$. Let $r = d(x, y)$. Consider the open sets given by $\mathcal{B}_{\frac{r}{3}}(x)$ and $\mathcal{B}_{\frac{r}{3}}(y)$. Suppose $z \in \mathcal{B}_{\frac{r}{3}}(x) \cap \mathcal{B}_{\frac{r}{3}}(y)$. Then we have that $d(x, z) < \frac{r}{3}$ and $d(y, z) < \frac{r}{3}$. It follows that

$$\begin{aligned} r &= d(x, y), \\ &\leqslant d(x, z) + d(y, z), \\ &< \frac{r}{3} + \frac{r}{3}, \\ &= \frac{2r}{3}. \end{aligned} \tag{5.30}$$

We found a contradiction. Thus, the hypothesis that there is $z \in \mathcal{B}_{\frac{r}{3}}(x) \cap \mathcal{B}_{\frac{r}{3}}(y)$ and we may conclude (M, d) is Hausdorff. ■

Lemma 90:

Let (M, d) *be a complete metric space. Let* $N \subseteq M$ *and let us consider* $(N, d\big|_N)$ *as a metric subspace of* (M, d)*. For simplicity, we shall write* $d\big|_N$ *just as* d*.* (N, d) *is complete if, and only if,* N *is a closed set.* □

Proof:

⇒: Let us assume N is a closed set. Then it holds that $N = \overline{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of N. Since it is Cauchy and (M, d) is complete, there is some $x \in M$ such that $x_n \to x$. Notice that $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood O of x. However, we know from Theorem 43 that $x \in \overline{N} \Leftrightarrow O \cap N \neq 0$ for every neighborhood O of x. Since given a neighborhood O of x there is $n_0 \in \mathbb{N}$ such that $x_n \in O, \forall\, n > n_0$ and we already now that $x_n \in N, \forall\, n \in \mathbb{N}$, we see that $O \cap N \neq 0$ for every neighborhood O of x. Therefore, $x \in \overline{N} = N$, and we see that every Cauchy sequence in N admits a limit in N.

⇐: Suppose every Cauchy sequence of elements of N is convergent. Let $x \in \overline{N}$. Then Theorem 43 states $x \in \overline{N} \Leftrightarrow O \cap N \neq 0$ for every neighborhood O of x. Let us pick, in particular, the open balls centered at x and with radius $\frac{1}{n}$. $\forall\, n \in \mathbb{N}$, let $x_n \in \mathcal{B}_{\frac{1}{n}}(x) \cap N$.

This is a Cauchy sequence, for $d(x_n, x_m) \leqslant d(x_n, x) + d(x_m, x) < \frac{1}{n} + \frac{1}{m}$. Given $\epsilon > 0$, we can pick $n_0 \in \mathbb{N}$ respecting $n_0 > \frac{2}{\epsilon}$ and it holds that $d(x_n, x_m) < \epsilon, \forall\, n, m > n_0$.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and (N, d) is complete, we have that $\lim x_n \in N$. Notice that every metric space is a Hausdorff space (Lemma 89) and every sequence in a Hausdorff space admits at most one limit (Theorem 53). Thus, we may conclude that $\lim x_n = x$, and it follows that $x \in N, \forall\, x \in \overline{N}$. Since it always holds that $N \subseteq \overline{N}$, we conclude that $N = \overline{N}$ and, therefore, that N is closed. ■

Proposition 91:
Let (X, τ) be a topological space. $BC(X)$ is a closed subspace of $B(X)$ under the uniform metric. Furthermore, $BC(X)$ is complete. □

Proof:
Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $BC(X)$. Since $B(X)$ is complete, we know there is $f \in B(X)$ such that $f_n \to f$ in the uniform metric. If we prove that $f \in C(X)$, then we have that $f \in BC(X)$ and may conclude that $BC(X)$ is complete (and closed, due to Lemma 90).

Let $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}, \forall\, n > n_0$. Let $n > n_0$ and $x \in X$. We know f_n is continuous at x. Thus, given a neighborhood O of $f_n(x)$, we have a neighborhood U of x such that $U \subseteq f_n^{-1}(O)$. In particular, we might consider $O = \mathcal{B}_{\frac{\epsilon}{3}}(f_n(x))$ (where the metric considered is the Euclidean metric in $\mathbb{C}$). Thus, we see $\forall\, y \in U$ it holds that $|f_n(y) - f_n(x)| < \frac{\epsilon}{3}$. We have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)|, \\ &\leqslant |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|, \\ &\leqslant \epsilon. \end{aligned} \tag{5.31}$$

Now, notice that $x \in U \subseteq f^{-1}\left(\mathcal{B}_\epsilon(f(x))\right)$. Thus, f is continuous at x. ■

6 Countability and Separation Axioms

You might have noticed by now that Topology is quite a general theory. Indeed, it is too general for usual physical purposes and has too few axioms to generate interesting results. This leads us to the restrict ourselves to a smaller class of topological spaces. In the process, we do lose generality, but also are able to obtain more results and work more intensely with the spaces that do interest us.

Two categories of axioms commonly used are the countability and separation axioms.

Definition 92 [First Axiom of Countability]**:**

A topological space (X, τ) is said to satisfy the *first axiom of countability*, or to be *first-countable*, whenever, $\forall\, x \in X$, there is a countable neighborhood basis for τ at x. ♠

Proposition 93:

Let (X, τ) *be a first-countable topological space. Then,* $\forall\, x \in X$, *there is a neighborhood base* $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ *such that* $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i, \forall\, i$. □

Proof:

Let $x \in X$. Since (X, τ) is first-countable, there is a countable neighborhood base $\left\{O_j\right\}_{j=1}^{\infty}$ for τ at x. We may then define another neighborhood basis $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ through

$$\mathcal{B}_i = \bigcap_{j=1}^{i} O_j.$$

Notice that $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ is indeed a neighborhood basis for τ at x: since $x \in O_j, \forall\, j$, it follows that $x \in \mathcal{B}_i, \forall\, i$. If $x \in O \in \tau, \exists\, j; x \in O_j$. Therefore, $x \in \mathcal{B}_j$. ■

Proposition 94:

Let (X, d) *be a metric space. Then* (X, d) *is first-countable.* □

Proof:

Let $x \in X$. The collection $\mathfrak{N} = \left\{\mathcal{B}_{\frac{1}{n}}(x); n \in \mathbb{N}\right\}$ is a countable neighborhood basis for the topology generated by d at x. Indeed, $x \in \mathcal{B}_{\frac{1}{n}}(x), \forall\, n \in \mathbb{N}$. Furthermore, if O is open and $x \in O$, then there is some $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x) \subseteq O$. The Archimedean property of the real line guarantees the existence of $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$, and we then have $x \in \mathcal{B}_{\frac{1}{n_0}}(x) \subseteq \mathcal{B}_\epsilon(x) \subseteq O$. This proves the result. ■

Definition 95 [Second Axiom of Countability]**:**

A topological space (X, τ) is said to satisfy the *second axiom of countability*, or to be *second-countable*, whenever there is a countable basis for τ on X. ♠

Proposition 96:

Let (X, τ) *be a second-countable topological space. Let* $Y \subseteq X$ *and let* τ_Y *be the relative topology on* Y. (Y, τ_Y) *is second-countable.* □

Proof:

Let $\mathfrak{B}$ be a countable basis for (X, τ). Then $\mathfrak{B}_Y \equiv \{\mathcal{B} \cap Y; \mathcal{B} \in \mathfrak{B}\}$ is a basis for the relative topology in Y.

Indeed, since $\mathfrak{B}$ is a basis for (X, τ), it holds that $\forall\, y \in Y, \exists\, \mathcal{B} \in \mathfrak{B}; y \in \mathcal{B}$. Hence, $y \in \mathcal{B} \cap Y$.

Furthermore, $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. This implies in particular that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, y \in \mathcal{B}_1 \cap \mathcal{B}_2 \cap Y, \exists\, \mathcal{B}_3 \in \mathfrak{B}; y \in \mathcal{B}_3 \cap Y \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap Y$. Thus, $\mathfrak{B}_Y$ is a basis for a topology in Y.

We know that

$$\tau = \{O \subseteq X |\, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\}\,. \tag{6.1}$$

Also, we know

$$\begin{aligned}
\tau_Y &= \{(U \cap Y) \subseteq Y; U \in \tau\}\,, \\
&= \{(U \cap Y) \subseteq Y |\, \forall\, x \in U, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq U\}\,, \\
&= \{(U \cap Y) \subseteq Y |\, \forall\, y \in U \cap Y, \exists\, \mathcal{B} \in \mathfrak{B}; y \in (\mathcal{B} \cap Y) \subseteq (U \cap Y)\}\,, \\
&= \{U \subseteq Y |\, \forall\, y \in U, \exists\, \mathcal{B} \in \mathfrak{B}_Y; y \in \mathcal{B} \subseteq U\}\,.
\end{aligned} \tag{6.2}$$

This concludes the proof. ■

We shall also define a third countability axiom which exemplifies how bad terminology can get.

Definition 97 [Separable Space]**:**

A topological space (X, τ) is said to be *separable* whenever it has a countable dense subset, *id est*, whenever there is a countable set A such that $\overline{A} = X$. ♠

Proposition 98:

Let (X, τ) *be a second-countable topological space. Then* (X, τ) *is separable.* □

Proof:

Since (X, τ) is second-countable, there is a countable basis for τ in X, which we shall write $\{\mathcal{B}_i\}_{i=1}^{+\infty}$. With the Axiom of Choice, $\forall\, i$ we choose a point $x_i \in \mathcal{B}_i$. Consider now the set $A = \{x_i\}_{i=1}^{+\infty}$. Notice that $\overline{A}^c$ is an open set, and thus there is a family of indexes Λ such that $\overline{A}^c = \bigcup_{\lambda \in \Lambda} \mathcal{B}_{i_\lambda}$.

Let us suppose that $\Lambda \neq \varnothing$ and fix $\lambda \in \Lambda$. We know that $x_{i_\lambda} \in \mathcal{B}_{i_\lambda}$ and thus $x_{i_\lambda} \in \overline{A}^c$. However, $x_{i_\lambda} \in A \subseteq \overline{A}$, which means we have reached a contradiction. We must have $\Lambda = \varnothing$, and it follows that $\overline{A}^c = \varnothing$. Therefore, $\overline{A} = X$. Since A is countable, we have shown the existence of a countable dense subset of X, which means (X, τ) is separable. ■

Definition 99 [Lindelöf Space]**:**

Let (X, τ) be a topological space. X is said to be a *Lindelöf space* if, and only if, every open cover of X admits a countable subcover. ♠

Proposition 100:

Every second-countable space is a Lindelöf space. □

Proof:

Let (X, τ) be a second-countable space. Since it is second-countable, we know it possesses a countable basis, which we shall call $\mathfrak{B}$. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X, where Λ is an arbitrary family of indices.

Since, $\forall\, \lambda \in \Lambda, A_\lambda$ is open, we know all the A_λ are unions of elements of $\mathfrak{B}$.

Let us define

$$\mathfrak{B}' \equiv \{\mathcal{B} \in \mathfrak{B} | \exists \lambda \in \Lambda; \mathcal{B} \subseteq A_\lambda\}. \tag{6.3}$$

$\mathfrak{B}'$ covers the entire space, for the A_λ are unions of elements of $\mathfrak{B}'$ and they cover the entire space. Since $\mathfrak{B}' \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is countable, $\mathfrak{B}'$ is countable as well.

Since $\mathfrak{B}'$ is countable, we may now enumerate its elements. Let $\mathcal{B}\colon \mathbb{N} \to \mathfrak{B}'$ be a surjective function such that $n \mapsto \mathcal{B}_n$. Then, for each $n \in \mathbb{N}$, we may pick $\lambda_n \in \Lambda; \mathcal{B}_n \subseteq A_{\lambda_n}$. The existence of λ_n is guaranteed by the definition of $\mathfrak{B}'$. Notice now that

$$\bigcup_{n=1}^{+\infty} \mathcal{B}_n = X,$$
$$\therefore \mathcal{B}_n \subseteq A_{\lambda_n}, \forall n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{+\infty} A_{\lambda_n} = X. \tag{6.4}$$

Thus, $\{A_{\lambda_n}\}_{n\in\mathbb{N}}$ is a countable subcover of $\{A_\lambda\}_{\lambda\in\Lambda}$. ■

The next result illustrates how these extra axioms can be useful when dealing with more concrete concepts such as the convergence of sequences.

Proposition 101:

Let (X, τ) be a first-countable topological space and let $A \subseteq X$. Then $x \in \overline{A}$ if, and only if, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A that converges to x. □

Proof:

- $\Rightarrow$: Let us assume $x \in \overline{A}$. From Theorem 43, we know that every neighborhood of x intersects A. From Proposition 93, we know there is a countable neighborhood basis $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ for τ in x satisfying $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$. Since $\{\mathcal{B}_i\}_{i=1}^{+\infty} \subseteq \tau$, we may conclude that $A \cap \mathcal{B}_i \neq \varnothing, \forall i$. We may then use the Axiom of Choice to pick, $\forall i$, $x_i \in A \cap \mathcal{B}_i$. Since the neighborhood basis is countable, this does define a sequence. Notice that $x_i \in \mathcal{B}_i \subseteq \mathcal{B}_j, \forall j < i$.

  We remain to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ does converge to x, *id est*, we remain to prove that $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood of x. In order to do so, let O be such a neighborhood. Since $x \in O \in \tau$, we know that $\exists i; \mathcal{B}_i \subseteq O$ (for $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ is a neighborhood basis for τ in x). Thus, $x_i \in O$. Since $x_j \in \mathcal{B}_j \subseteq \mathcal{B}_i \subseteq O, \forall j > i$, we see that $(x_n)_{n\in\mathbb{N}}$ is eventually in O. As the argument holds for every neighborhood O of x, we conclude that $x_n \to x$, as desired.

- $\Leftarrow$: Suppose now that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A with $x_n \to x$. Therefore, if O is a neighborhood of x, $\exists n_0 \in \mathbb{N}; x_n \in O, \forall n \geqslant n_0$. Since $x_n \in A, \forall n$, this means that $A \cap O \neq \varnothing$ for every neighborhood O of x. Thus, it follows from Theorem 43 that $x \in \overline{A}$, as desired. ■

Theorem 102:

Let (M, d) be a metric space. A set A is dense in the sense of metric spaces, id est, $\forall \epsilon > 0, \forall x \in M, \exists p \in A; p \in \mathcal{B}_\epsilon(x)$ *if, and only if, it is dense in the sense of topological spaces,* id est, $\overline{A} = M$. □

Proof:

From Propositions 94 and 101, we know $x \in \overline{A}$ if, and only if, there is some sequence of elements of A converging to x.

Suppose A is dense in the sense of metric spaces. Then, given $x \in X$, we know $\forall \epsilon > 0, \exists p \in A; p \in \mathcal{B}_\epsilon(x)$. Thus, $\forall n \in \mathbb{N}, \exists x_n \in A; x_n \in \mathcal{B}_{\frac{1}{n}}(x)$. This is a sequence with $x_n \to x$, and thus $x \in \overline{A}$.

Suppose now that $\overline{A} = X$. Then, $\forall x \in X$, there is some sequence of elements of X with $x_n \to x$. Thus, given x, we know there is a sequence x_n of elements of A such that $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}; d(x, x_n) < \epsilon$, *id est*, $\forall \epsilon > 0, \exists p \in A; p \in \mathcal{B}_\epsilon(x)$.

This concludes the proof. ■

Proposition 103:

Let (M, d) be a metric space. The following statements are equivalent:

i. *it is a Lindelöf space;*

ii. *it is separable;*

iii. *it is second-countable.* □

Proof:

i. $\Rightarrow$ ii.: For each $n \in \mathbb{N}$, consider the open cover of M given by $\left\{\mathcal{B}_{\frac{1}{n}}(x)\right\}_{x \in M}$. Since (M, d) is a Lindelöf space, these open covers admit countable subcovers given by the collections $\left\{\mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right)\right\}_{m \in \mathbb{N}}$. Consider the set $A = \left\{x_m^{(n)}\right\}_{n, m \in \mathbb{N}}$, which is countable, for it is the countable union of countable sets. Let $x \in M$ and $\epsilon > 0$. The Archimedean property of the real numbers allows us to find $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Since $\left\{\mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right)\right\}_{m \in \mathbb{N}}$ covers X, there is $m \in \mathbb{N}$ such that $x \in \mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right) \subseteq \mathcal{B}_\epsilon\left(x_m^{(n)}\right)$. Therefore, A is dense in M in the sense of metric spaces. Theorem 102 guarantees $\overline{A} = M$. Since A is countable, (M, d) is separable.

ii. $\Rightarrow$ iii.: Since (M, d) is separable, there is some countable subset A such that $\overline{A} = M$. We can consider the collection $\mathfrak{B}$ defined by

$$\mathfrak{B} = \left\{\mathcal{B}_{\frac{1}{n}}(x); n \in \mathbb{N}, x \in A\right\}. \tag{6.5}$$

$\mathfrak{B}$ is a countable basis for the metric topology on (M, d). It is guaranteed to cover X due to the fact that A is dense in the sense of metric spaces (as per Theorem 102), so $\forall x \in M$ and $\forall n \in \mathbb{N}$, there is $p \in A; x \in \mathcal{B}_{\frac{1}{n}}(p)$. The fact that

$$\forall \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \tag{6.6}$$

can be proven from the triangle inequality and from the Archimedean property. The topology generated by $\mathfrak{B}$ is indeed the metric topology on (M, d), for the elements of $\mathfrak{B}$ are the open balls with respect to d.

iii. $\Rightarrow$ i.: Proposition 100. ■

Example:

Let $X = [0,1]$ and let $\tau = \{O \subseteq X; O^c \text{ is countable}\}$. τ defines a topology* on X. Let $A = [0,1)$. Since A is not countable, $\{1\}$ is not open and, as a consequence, A is not closed. Therefore, $\overline{A} = X$. I claim there are no sequences of elements of A with $x_n \to 1$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of A. The set $B = \{x_n; n \in \mathbb{N}\}$ is countable, and therefore B^c is an open set. Since $1 \notin A$ and $B \subseteq A$, $1 \in B^c$. Therefore, B^c is an open set containing 1, a neighborhood of 1, that does not contain any elements of the sequence $(x_n)_{n\in\mathbb{N}}$. Therefore, is does not hold that $(x_n)_{n\in\mathbb{N}}$ is eventually in any neighborhood of 1 and it follows that $x_n \not\to 1$. Furthermore, since no elements of $(x_n)_{n\in\mathbb{N}}$ are in B, 1 isn't even a cluster point of $(x_n)_{n\in\mathbb{N}}$. ♥

Let us now pay attention to the separation axioms. We have already presented one of them in Definition 52, and we shall define it once again so all the axioms can be together in a single definition. You may notice some resemblances between them.

Definition 104 [Separation Axioms]**:**

Let (X,τ) be a topological space. If it has the property T_j, it is said to be a T_j space or that the topology on X is T_j. We suppose $x, y \in X$. The axioms read:

T_0: If $x \neq y, \exists\, O \in \tau; x \in O, y \notin O$ *or* $x \notin O, y \in O$.

T_1: If $x \neq y, \exists\, O \in \tau; x \in O, y \notin O$.

T_2: If $x \neq y, \exists\, O, U \in \tau; x \in O, y \in U, O \cap U = \varnothing$.

T_3: The space is T_1 and for every closed set $F \subseteq X$ and $\forall\, x \in F^c, \exists\, O, U \in \tau; x \in O, F \subseteq U, O \cap U = \varnothing$.

$T_{3\frac{1}{2}}$: The space is T_1 and for every closed set $F \subseteq X$ and $\forall\, x \in F^c$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0, \forall\, y \in F$.

T_4: The space is T_1 and for every disjoint closed sets $F, G \subseteq X, \exists\, O, U \in \tau; F \subseteq O, G \subseteq U, O \cap U = \varnothing$.

A space with the T_2 property is also called a *Hausdorff* space (and T_2 is also called the *Hausdorff property*). A space with the T_3 property is also called a *regular* space. A space with the $T_{3\frac{1}{2}}$ property is also called a *completely regular* space or a *Tychonoff* space. A space with the T_4 property is also called a *normal* space. Some authors might not require for a space to be T_1 in order to be regular, completely regular or normal. ♠

Remark:

The definition for a Tychonoff space might seem awkward right now, and the fact that it is associated with the $T_{3\frac{1}{2}}$ property is not exactly helpful. The truth is this axiom is intermediate between T_3 and T_4 and the mysteriousness behind this definition shall vanish once we study Urysohn's Lemma (Lemma 109). ♣

Proposition 105:

Let (X,τ) be a topological space. (X,τ) is T_1 if, and only if, $\{x\}$ is closed $\forall\, x \in X$. □

*Named the cocountable topology

Proof:

- $\Leftarrow$: Suppose $\{x\}$ is closed $\forall\, x \in X$. Let $x, y \in X, x \neq y$. Then $\{y\}^c$ is an open set such that $x \in \{y\}^c$, but $y \notin \{y\}^c$. In a similar way, $x \notin \{x\}^c$, but $y \in \{x\}^c$. Thus, (X, τ) is T_1.

- $\Rightarrow$: Suppose (X, τ) is T_1. Let $x \in X$. We want to prove that $\{x\}$ is a closed set.

  We know that $\forall\, y \in X, y \neq x, \exists\, O_y \in \tau; y \in O_y, x \notin O_y$ (for (X, τ) is T_1). Since the arbitrary union of open sets is an open set, $O \equiv \bigcup_{y \in X \setminus \{x\}} O_y$ is an open set. Notice that, since $y \in O_y, x \notin O_y, \forall\, y \in X \setminus \{x\}$, we have that $O = X \setminus \{x\}$. Since O is an open set, $O^c = \{x\}$ is a closed set. This concludes the proof. ∎

Proposition 106:

Let (X, τ) be a Hausdorff topological space. Let $Y \subseteq X$ and let τ_Y be the relative topology on Y. (Y, τ_Y) is Hausdorff. □

Proof:

Let $x, y \in Y, x \neq y$. We want to prove there are open sets $O, U \in \tau_Y$ such that $x \in O, y \in U, O \cap U = \varnothing$.

Since (X, τ) is Hausdorff and $x, y \in Y \subseteq X$, we see there are open sets $O_X, U_X \in \tau$ such that $x \in O_X, y \in U_X, O_X \cap U_X = \varnothing$. The sets $O = O_X \cap Y$ and $U = U_X \cap Y$ are open in Y and satisfy $x \in O, y \in U, O \cap U = \varnothing$. This concludes the proof. ∎

Proposition 107:

Let (X, τ) be a topological space. Let $\Delta = \{(x, y) \in X \times X; x = y\}$. (X, τ) is Hausdorff if, and only if, Δ is closed in the product topology. □

Proof:

Suppose (X, τ) is Hausdorff. We want to prove that given any $(x, y) \in \overline{\Delta}$, it holds that $x = y$. Due to Theorem 43, we know that $(x, y) \in \overline{\Delta}$ if, and only if, $O_x \times O_y$ intersects Δ for all $O_x, O_y \in \tau$ with $x \in O_x, y \in O_y$, for $\mathfrak{B} = O \times U; O, U \in \tau$ is a basis for the product topology.

Let $(x, y) \in \overline{\Delta}$. Suppose $x \neq y$. The Hausdorff property ensures the existence of $O, U \in \tau$ such that $x \in O, y \in U, O \cap U = \varnothing$. However, we see that $(O \times U) \cap \Delta \neq \varnothing$. Thus, there is $z \in X$ such that $(z, z) \in O \times U$, *id est*, $z \in O \cap U$. This contradicts the fact that $O \cap U \neq \varnothing$, proving the assumption $x \neq y$ is false. Hence, $(x, y) \in \overline{\Delta} \Rightarrow x = y$, proving $\overline{\Delta} \subseteq \Delta$ and, as a consequence, that Δ is closed.

Suppose now that Δ is closed. It follows that $\overline{\Delta} = \Delta$, and hence $(x, y) \in \overline{\Delta} \Rightarrow x = y$. Theorem 43 guarantees then that, if $x \neq y$, there are $O, U \in \tau$ with $x \in O, y \in \tau$ such that $O \times U$ does not intersect Δ.

Pick $x, y \in X$. We know there are $O, U \in \tau$ with the properties that $x \in O, y \in U$ and $(w, z) \in O \times U \Rightarrow w \neq z, \forall (w, z) \in O \times U$. Thus, $w \in O \Rightarrow w \notin U$ and $z \in U \Rightarrow z \notin O$. Hence, $O \cap U = \varnothing$, proving (X, τ) is Hausdorff. ∎

7 Urysohn's Lemma

We might now start studying some consequences of the countability and separation axioms. Sometimes, even regular spaces admit only the constant functions as continuous functions, but we shall see that normal spaces always have a good amount of continuous functions.

Lemma 108:

Let (X, τ) *be a normal topological space. Let* $A, B \subseteq X$ *be a pair of disjoint closed sets. Consider the set of dyadic rational numbers in the interval* $[0, 1]$, *given by*

$$\Delta = \left\{ \frac{k}{2^n}; n \in \mathbb{N}^*, k \in \{i\}_{i=0}^{2^n} \right\}. \tag{7.1}$$

There is a collection $\{U_r\}_{r \in \Delta}$ *of open sets in* X *satisfying* $A \subseteq U_r \subseteq B^c, \forall\, r \in \Delta$ *with* $r < s \Rightarrow \overline{U}_r \subseteq U_s$. □

Proof:

We shall proceed by induction.

For $n = 1$, we have $r = 0$ and $r = 1$. Since A and B are disjoint closed sets in a normal space, there is a pair of disjoint open sets O and U satisfying $A \subseteq O$ and $B \subseteq U$. Lets us define $U_0 \equiv O$ and $U_1 \equiv B^c$. Indeed, since $O \cap U = \varnothing$, it holds that $O \cap B = \varnothing$ and, as a consequence, $O \subseteq B^c$. Thus, $A \subseteq U_0 \subseteq B^c$. We still must prove that $\overline{U}_0 \subseteq U_1 = B^c$.

$\overline{U}_0 \subseteq B^c$ if, and only if, $\overline{U}_0 \cap B = \varnothing$. Let $x \in B$. We know from Theorem 43 that $x \in \overline{U}_0$ if, and only if, $U_0 \cap O_x \neq \varnothing$, for every neighborhood O_x of x. However, since $B \subseteq U$ and U is an open set satisfying $U \cap O = \varnothing$, U is a neighborhood of x that doesn't intersect $O = U_0$. Thus, $\overline{U}_0 \cap B = \varnothing$ and it holds that $\overline{U}_0 \subseteq U_1$.

We must now prove the inductive step. Suppose there are open sets U_r satisfying our desires for every rational of the form $r = \frac{k}{2^{n-1}}$. We want to prove it holds for rationals of the form $r = \frac{k}{2^n}$ as well. If k is an even number, then it holds that $r = \frac{l}{2^{n-1}}$, with $l = \frac{k}{2}$, and it is done by hypothesis. Let us bother with k odd then. We know that for $r_0 = \frac{k-1}{2^n}$ and $r_1 = \frac{k+1}{2^n}$ there are open sets U_{r_0} and U_{r_1} with $A \subseteq U_{r_0} \subseteq \overline{U}_{r_0} \subseteq U_{r_1} \subseteq B^c$.

Consider now the closed sets $\overline{U}_{r_0}$ and $U^c_{r_1}$. Since (X, τ) is normal, there are open sets O_0 and O_1 such that $\overline{U}_{r_0} \subseteq O_0$, $U^c_{r_1} \subseteq O_1$ and $O_0 \cap O_1 = \varnothing$. We already see that $A \subseteq U_{r_0} \subseteq \overline{U}_{r_0} \subseteq O_0 \subseteq U_{r_1} \subseteq B^c$ (since $U^c_{r_1} \subseteq O_1$ and $O_0 \cap O_1 = \varnothing$, it must hold that $O_0 \subseteq U_{r_1}$). Let us now show that $\overline{O}_0 \subseteq U_{r_1}$.

Let $x \in U^c_{r_1}$. If $x \in \overline{O}_0$, then it must hold that $O_x \cap O_0 \neq \varnothing$ for every neighborhood O_x of x. However, since $x \in U^c_{r_1} \in O_1$ and O_1 is an open set satisfying $O_1 \cap O_0$, there is a neighborhood of x that doesn't intersect O_0. Thus, $x \notin \overline{O}_0$. We might conclude that $\overline{O}_0 \subseteq U_{r_1}$.

Finally, we might now set $U_r \equiv O_0$, for $r = \frac{k}{2^n}$, and the proof is complete. ■

Lemma 109 [Urysohn's Lemma]**:**

Let (X, τ) *be a normal topological space. Let* A, B *be a pair of closed sets in* X. *Then there is a continuous function* $f\colon X \to [0, 1]$ *satisfying* $f(A) = \{0\}$ *and* $f(B) = \{1\}$. □

Proof:

Let Δ be the set of dyadic rational numbers in the interval $[0, 1]$. Due to Lemma 108, we know that there is a collection $\{U_r\}_{r \in \Delta}$ of open sets in X satisfying $A \subseteq U_r \subseteq B^c, \forall\, r \in \Delta$

with $r < s \Rightarrow \overline{U}_r \subseteq U_s$. Let us define a function f by

$$\begin{cases} f(x) = 0, \forall x \in U_0, \\ f(x) = \sup\limits_{t \notin U_r} r, \forall x \notin U_0. \end{cases} \tag{7.2}$$

Since $A \subseteq U_0$ and $f(U_0) = \{0\}$, it holds that $f(A) = \{0\}$. Furthermore, since $x \in B \Rightarrow x \notin U_r, \forall r \in [0,1]$, it holds that $f(B) = \{1\}$. Furthermore, it is simple to see that $f(X) \subseteq [0,1]$. We now must prove that f is indeed a continuous function.

Let $\alpha \in [0,1]$. Notice that $f(x) < \alpha$ if, and only if, $x \in U_r$ for some $r < \alpha$, *id est*, if, and only if, $x \in \bigcup_{r<\alpha} U_r$. Thus, $f^{-1}\big((\alpha, +\infty)\big) = \bigcup_{r<\alpha} U_r$, which is an arbitrary union of open sets, and therefore is itself open.

On the other hand, if we let $\beta \in [0,1]$, $f(x) > \beta$ if, and only if, $x \in U_r^c$ for some $r > \beta$. However, since $\overline{U}_s \subseteq U_r$ for very $s < r$, this happens if, and only if, $x \in \overline{U}_s^c$ for some $s > \beta$, *id est*, if, and only if, $x \in \bigcup_{s>\beta} \overline{U}_s^c$. Thus, $f^{-1}\big((-\infty, \beta)\big) = \bigcup_{s>\beta} \overline{U}_s^c$, which is an arbitrary union of open sets, and therefore it is open as well.

Finally, notice that $f^{-1}\big((\alpha, \beta)\big) = f^{-1}\big((\alpha, +\infty)\big) \cap f^{-1}\big((-\infty, \beta)\big)$, which is a finite intersection of open sets, and thus an open set itself. Since the intervals comprise a basis for the standard topology on $\mathbb{R}$ and the preimage of the intervals are always open under f, it holds that f is indeed continuous. ■

Corollary 110:

Let (X, τ) be a normal topological space. Let A, B be a pair of closed sets in X. Let $a, b \in \mathbb{R}$, $a < b$. Then there is a continuous function $f\colon X \to [a, b]$ satisfying $f(A) = \{a\}$ and $f(B) = \{b\}$. □

Proof:

Due to Urysohn's Lemma, we know there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Since we may add and multiply continuous functions by other continuous functions without altering their continuity, let us consider the continuous function given by $g(x) = (b - a)f(x) + a$. Notice that $g(A) = \{a\}$ and $g(B) = \{b\}$. This concludes the proof. ■

Scholium:

We shall also refer to Corollary 110 as Urysohn's Lemma. ♣

Theorem 111 [Tietze Extension Theorem]**:**

Let (X, τ) be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to [a, b]$ be a continuous function. There is a continuous function $F\colon X \to [a, b]$ such that $F(x) = f(x), \forall x \in A$. □

Proof:

Let us first bother with the case $f\colon A \to [a, b]$. The remaining cases shall be proved as consequences. We can assume $[a, b] = [0, 1]$. Indeed, we would be able to extend $\frac{f-a}{b-a}$ to a function $\frac{F-a}{b-a}$ and F is now the function we were looking for. Thus, we assume for simplicity that $[a, b] = [0, 1]$.

I claim there is a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ on X satisfying $0 \leqslant g_n(x) \leqslant \frac{2^{n-1}}{3^n}, \forall x \in X$, and $0 \leqslant f(x) - \sum_{k=1}^{n} g_k(x) \leqslant \left(\frac{2}{3}\right)^n, \forall x \in A$.

Let us consider the sets $B = f^{-1}\left(\left[0, \frac{1}{3}\right]\right)$ and $C = f^{-1}\left(\left[\frac{2}{3}, 1\right]\right)$. These sets are closed as subsets of A (for f is continuous and closed intervals are closed in the standard topology of the real line). Since A is closed in X, B and C are closed in X (Corollary 28). Since (X, τ) is a normal space, Urysohn's Lemma guarantees the existence of a function $g_1 : X \to [0, \frac{1}{3}]$ with $g_1(B) = \{0\}$ and $g_1(C) = \left\{\frac{1}{3}\right\}$. As a consequence, $0 \leqslant f(x) - g_1(x) \leqslant \frac{2}{3}, \forall\, x \in A$.

Let us now consider the continuous function $h(x) = f(x) - g_1(x), \forall\, x \in A$. Notice that $h : A \to \left[0, \frac{2}{3}\right]$. We (re-)define $B = h^{-1}\left(\left[0, \frac{2^{2-1}}{3^2}\right]\right)$ and $C = h^{-1}\left(\left[\frac{2^2}{3^2}, \frac{2^{2-1}}{3^{2-1}}\right]\right)$. Once again it holds that B and C are closed sets in X and Urysohn's Lemma guarantees the existence of a continuous function $g_2 : X \to \left[0, \frac{2^{2-1}}{3^2}\right]$ with $g_2(B) = \{0\}$ and $g_2(C) = \left\{\frac{2^{2-1}}{3^2}\right\}$. Now we have that $0 \leqslant f(x) - g_1(x) - g_2(x) \leqslant \left(\frac{2}{3}\right)^2$.

In general, we consider the continuous function $h(x) = f(x) - \sum_{k=1}^{n-1} g_k(x), \forall\, x \in A$. It holds that $h : A \to \left[[0, \left(\frac{2}{3}\right)^{n-1}]\right]$. We (re-)define $B = h^{-1}\left(\left[0, \frac{2^{n-1}}{3^n}\right]\right)$ and $C = h^{-1}\left(\left[\frac{2^n}{3^n}, \frac{2^{n-1}}{3^{n-1}}\right]\right)$. Urysohn's Lemma guarantees the existence of a continuous function $g_n : X \to \left[0, \frac{2^{n-1}}{3^n}\right]$ with $g_n(B) = \{0\}$ and $g_n(C) = \left\{\frac{2^{n-1}}{3^n}\right\}$. We finally have that $0 \leqslant f(x) - \sum_{k=1}^{n} g_k(x) \leqslant \left(\frac{2}{3}\right)^n$.

Let we define $F_n(x) = \sum_{k=1}^{n} g_k(x), \forall\, x \in X$. Let d denote the uniform metric on $BC(X)$. Let now $\epsilon > 0$. We know from Real Analysis that there is $n_0 \in \mathbb{N}$ such that, $\forall\, n, m > 0$ (let us suppose, without any loss of generality, that $m \geqslant n$),

$$\begin{aligned} d(F_n, F_m) &\leqslant d(F_n, 0) + d(F_m, 0), \\ &= \sup_{x \in X} |F_n(x)| + \sup_{x \in X} |F_m(x)|, \\ &\leqslant 2\left(\frac{2}{3}\right)^n, \\ &< \epsilon. \end{aligned} \tag{7.3}$$

Therefore, $(F_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $BC(X)$, which is a complete metric space when equipped with the uniform metric (Proposition 91). Therefore, there is a function $F \in BC(X)$ such that $F(x) = \sum_{k=1}^{+\infty} g_k(x), \forall\, x \in X$.

Notice that $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in A, |f(x) - F_n(x)| \leqslant \left(\frac{2}{3}\right)^n < \epsilon$. Therefore, $F_n \to f$ and it holds that $F(x) = f(x), \forall\, x \in A$.

We still must prove that $F : X \to [0, 1]$. Since $g_k(x) \geqslant 0, \forall\, k \in \mathbb{N}^*, \forall\, x \in X$, of course $F(x) \geqslant 0, \forall\, x \in X$.

In order to force the extension to be limited to $[0, 1]$, we may define a new extension which surely respects that and use it as the extension. We already know that F is bounded (for $F \in BC(X)$), and thus we just need to re-scale it in a manner that preserves its continuity and without altering its value in A.

Consider the map $r : \mathbb{R}_+ \to [0, 1]$ given by

$$\{r(x) = x, \text{ if } x \leqslant 1, \quad r(x) = 1, \text{ if } x > 1. \tag{7.4}$$

This map is continuous when $\mathbb{R}_+$ and $[0, 1]$ are equipped with the relative topology (with respect to the standard topology in $\mathbb{R}$). Indeed, $r^{-1}([0, a)) = [0, a)$, which is open in $\mathbb{R}_+$. In a similar manner, $r^{-1}((a,1]) = (a,+\infty)$, which is open as well. Finally, $r^{-1}((a,b)) = (a,b)$, which is also open. These intervals are a basis in $[0,1]$ due to Lemma 24, and therefore r is continuous. As a consequence, the composition $F' \equiv r \circ F$ is continuous as well. Notice that $F(x) \in [0,1], \forall\, x \in A$, and thus F' is also an extension of f.

This concludes the proof. ■

Corollary 112:

Let (X,τ) be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to \mathbb{R}$ be a continuous function. There is a continuous function $F\colon X \to \mathbb{R}$ such that $F(x) = f(x), \forall\, x \in A$. □

Proof:

Since $\mathbb{R}$ and $(-1,1)$ are homeomorphic (Proposition 65), we might instead consider a function $f\colon A \subseteq (-1,1)$. The result will hold by composing the functions f and F with any homeomorphism between $\mathbb{R}$ and $(-1,1)$.

Due to the Tietze Extension Theorem, we know that there is a function $g\colon X \to [-1,1]$ which extends f continuously. We want to find a function $h\colon X \to (-1,1)$ which extends f.

With g given by the Tietze Extension Theorem, we define the set $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$. g is continuous and $\mathbb{R}$ is T_1*, and therefore it holds that D is closed. Since g extends f and $f(A) = (-1,1)$, it holds that $D \cap A = \varnothing$. Urysohn's Lemma then allows us to obtain a continuous function $\phi\colon X \to [0,1]$ with $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. We might now define $h(x) \equiv \phi(x)g(x)$. This is a product of continuous functions, and thus it continuous itself. Furthermore, if $x \in A$, $h(x) = \phi(x)g(x) = g(x) = f(x)$, and therefore h extends f. Notice that $h\colon X \to (-1,1)$, for $h(D) = \{0\}$ and $h(D^c) \in (-1,1)$. ■

Corollary 113:

Let (X,τ) be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to \mathbb{C}$ be a continuous function. There is a continuous function $F\colon X \to \mathbb{C}$ such that $F(x) = f(x), \forall\, x \in A$. □

Proof:

It suffices to consider real and imaginary parts separately, for a function is continuous if, and only if, the coordinate functions are continuous as well. We are now left with the same problem for real functions, which was solved in Corollary 112. ■

8 Nets

When dealing with Topology, the notion of sequence isn't always appropriate, due to the generality of the spaces we deal with. In order to study them, we first must define what is a directed system.

Definition 114 [Partial Orderings and Posets]**:**

Let $\prec$ be a relation on a set X. $\prec$ is said to be a *partial ordering* if, and only if, the following properties hold:

i. $\forall\, x \in X, x \prec x$ (reflexive);

ii. $\forall\, x,y,z \in X, x \prec y$ and $y \prec z \Rightarrow x \prec z$ (transitive);

*This follows from the fact that $\mathbb{R}$ is a metric space and we have already proven that every metric space is Hausdorff. The proof that every Hausdorff space is T_1 is straightforward.

iii. $\forall x, y \in X, x \prec y$ and $y \prec x \Rightarrow x = y$ (antisymmetric).

The pair $(X, \prec)$, where X is a set and $\prec$ is a partial ordering, is said to be a *partially ordered set* or a *poset*. ♠

Notation:

We might write $y \succ x$ to mean $x \prec y$. Both statements are equivalent. ♦

Definition 115 [Directed System]**:**

Let I be a set and let $\prec$ be a partial ordering on I. $(I, \prec)$ is said to be a *directed system*, or *directed set*, if, and only if, $\forall \alpha, \beta \in I, \exists \gamma \in I; \alpha \prec \gamma, \beta \prec \gamma$. ♠

Proposition 116:

Let (X, τ) be a topological space and let $x \in X$. Let $I = \{O \in \tau; x \in O\}$ and let $O \prec U$ if, and only if, $U \subseteq O$. $(I, \prec)$ is a directed system. □

Proof:

Since $O \subseteq O, \forall O \in I$, $\prec$ is reflexive.

If $O \prec U$ and $U \prec V$, then we know that $U \subseteq O$ and $V \subseteq U$ and it follows that $V \subseteq O$, *id est*, $O \prec V$. Thus, $\prec$ is transitive.

If $O \prec U$ and $U \prec O$, we see that $U \subseteq O$ and $O \subseteq U$. Therefore, $O = U$ and we see that $\prec$ is antisymmetric.

Let now $O, U \in I$. The set $O \cap U$ is open and, since $x \in O$ and $x \in U$, it holds that $x \in O \cap U$. Therefore, $O \cap U \in I$. Notice that $O \cap U \succ O$ and $O \cap U \succ U$. Therefore, it holds that $\forall O, U \in I, \exists V = O \cap U; O \prec V, U \prec V$. ■

Proposition 117:

Let $(I, \prec_I)$ and $(J, \prec_J)$ be directed systems. Consider the set $I \times J$ with the relation $(\alpha, \beta) \prec (\alpha', \beta') \Leftrightarrow \alpha \prec_I \alpha'$ and $\beta \prec_J \beta'$. $(I \times J, \prec)$ is a directed system. □

Proof:

Let $\alpha, \alpha' \in I, \beta, \beta' \in J$.

It is simple to show that the properties of a partial order are inherited from $\prec_I$ and $\prec_J$.

We know that there are $\gamma \in I$ and $\delta \in J$ with $\gamma \succ \alpha, \gamma \succ \alpha', \delta \succ \beta, \delta \succ \beta'$. Therefore, $(\alpha, \beta) \prec (\gamma, \delta)$ and $(\alpha', \beta') \prec (\gamma, \delta)$. ■

Definition 118 [Net]**:**

Let (X, τ) be a topological space and let $(I, \prec)$ be a directed system. A *net* in (X, τ) is a function $x \colon I \to X$. ♠

Notation:

As when dealing with sequences, instead of writing $x(\alpha)$ for the image of $\alpha \in I$ through a net x, it is usual to write simply x_α. It is also customary to write $(x_\alpha)_{\alpha \in I}$ for the net, instead of x. ♦

Remark:

Notice that $(\mathbb{N}, \leqslant)$, where $\leqslant$ denotes the usual order in the natural numbers, is a directed system, and as a consequence every sequence is a net. We are simply making the theory more general. ♣

We might now make the terms frequently and eventually, which we already defined when dealing with sequences, more general.

Definition 119 [Frequently and Eventually]**:**

Let $(I, \prec)$ be a directed system. Let $A \subseteq X$. Let $(x_\alpha)_{\alpha \in I}$ be a net. $(x_\alpha)_{\alpha \in I}$ is said to be *eventually* on A if, and only if, $\exists \, \beta \in I; x_\alpha \in A \, \forall \, \alpha \succ \beta$. $(x_\alpha)_{\alpha \in I}$ is said to be *frequently* in A if, and only if, $\forall \, \beta \in I, \exists \, \alpha \succ \beta; x_\alpha \in A$. ♠

Definition 120 [Cluster Point]**:**

Let (X, τ) be a topological space, $(I, \prec)$ be a directed system and $(x_\alpha)_{\alpha \in I}$ be a net of elements of X. A point $x \in X$ is said to be a *cluster point* of the net $(x_\alpha)_{\alpha \in I}$ with respect to the topology τ if, and only if, $(x_\alpha)_{\alpha \in I}$ is frequently in every neighborhood of x. ♠

Definition 121 [Limit Point]**:**

Let (X, τ) be a topological space, $(I, \prec)$ be a directed system and $(x_\alpha)_{\alpha \in I}$ be a net of elements of X. A point $x \in X$ is said to be an *limit point* (sometimes called simply *limit*) of the net $(x_\alpha)_{\alpha \in I}$ with respect to the topology τ if, and only if, $(x_\alpha)_{\alpha \in I}$ is eventually in every neighborhood of x. ♠

Notation:

If a net $(x_\alpha)_{\alpha \in I}$ has a point x as a limit point, we write $x_\alpha \to x$ and say that $(x_\alpha)_{\alpha \in I}$ converges to x. ♦

Definitions are always nice, but they seem pointless when ill-motivated. As a consequence, let us prove some results concerning nets as soon as possible in order to keep our interests clear.

Proposition 122:

Let (X, τ) be a topological space and let $A \subseteq X$. $x \in \overline{A}$ if, and only if, there is a net $(x_\alpha)_{\alpha \in I}$ of elements of A with $x_\alpha \to x$. □

Proof:

- ⇐: Suppose there is a net $(x_\alpha)_{\alpha \in I}$ of elements of A with $x_\alpha \to x$. This means that, for any neighborhood O of x, $\exists \, \beta \in I; x_\alpha \in O, \forall \, \alpha \succ \beta$. Since $x_\alpha \in A, \forall \, \alpha \in I$, this means that for any neighborhood O of x it holds that $O \cap A \neq \varnothing$. Thus, Theorem 43 implies that $x \in \overline{A}$.

- ⇒: Suppose $x \in \overline{A}$. We know from Proposition 116 that if we define $I = \{O \in \tau; x \in O\}$ and equip it with the partial ordering $O \prec U \Leftrightarrow U \subseteq O$, then $(I, \prec)$ is a directed system. Consider a net $(x_O)_{O \in I}$ such that $\forall \, O \in I, x_O \in O \cap A$.

  Is there such a net? Since $x \in \overline{A}$, every neighborhood of x - *id est*, every element of I - intersects A. Therefore, none of the sets $O \cap A$ is empty. I isn't empty either, for $X \in I$. Therefore, the Axiom of Choice guarantees the existence of such a net.

  I claim that this net satisfies $x_O \to x$. Let O be a neighborhood of x (*id est*, let $O \in I$). By construction, $x_O \in O$. Furthermore, $x_U \in U \subseteq O, \forall \, U \succ O$. Therefore, x is indeed a limit point of x_O. This concludes the proof. ■

Theorem 123:

Let (X,τ_X) and (Y,τ_Y) be topological spaces and $f\colon X \to Y$ be a function. f is continuous at $x \in X$ if, and only if, the net $(f(x_\alpha))_{\alpha\in I}$ converges to $f(x)$ for every net $(x_\alpha)_{\alpha\in I}$ converging to x. □

Proof:

- ⇒: Suppose f is continuous at x. Let $(x_\alpha)_{\alpha\in I}$ be a net converging to x. This means that for every neighborhood O of x, $\exists\, \beta \in I; x_\alpha \in O, \forall\, \alpha \succ \beta$.

  Let O be a neighborhood of $f(x)$. Since f is continuous, there is some open U with $x \in U \subseteq f^{-1}(O)$. Therefore, $\exists\, \beta \in I; x_\alpha \in U, \forall\, \alpha \succ \beta$. Notice that if $x_\alpha \in U \subseteq f^{-1}(O)$, then $f(x_\alpha) \in O$. Therefore, given a neighborhood O of $f(x)$, $\exists\, \beta \in I; f(x_\alpha) \in O, \forall\, \alpha \succ \beta$. Thus, $f(x_\alpha) \to x$.

- ⇐: We shall prove the contrapositive affirmation. Suppose f is not continuous at x. Then there is some neighborhood O of $f(x)$ for which there are no neighborhoods U of x satisfying $x \in U \subseteq f^{-1}(O)$. Therefore, $x \notin \overset{\circ}{f^{-1}(O)}$, *id est*, $x \in \overline{f^{-1}(O^c)}$. Proposition 122 guarantees that there is a net $(x_\alpha)_{\alpha\in I}$ of elements of $f^{-1}(O^c)$ converging to x. However, since $x_\alpha \in f^{-1}(O^c), \forall\, \alpha \in I$, it holds that $f(x_\alpha) \in O^c, \forall\, \alpha \in I$. Therefore, no element of the net $(f(x_\alpha))_{\alpha\in I}$ is ever in O, which is a neighborhood of $f(x)$. Therefore, we have found a net converging to x, but whose image does not converge to $f(x)$. ■

Proposition 124:

Let (X,τ) be a topological space. (X,τ) is Hausdorff if, and only if, every net $(x_\alpha)_{\alpha\in I}$ in X admits at most one limit point. □

Proof:

- ⇒: Assume (X,τ) is a Hausdorff space. Suppose $x \in X$ is a limit point for $(x_\alpha)_{\alpha\in I}$. Let $y \in X, y \neq x$. Since (X,τ) is a Hausdorff space, there are two disjoint open sets O, U satisfying $x \in O$, $y \in U$. Since $x_\alpha \to x$, we know there is $\beta \in I; x_\alpha \in O, \forall\, \alpha \succ \beta$. Since $O \cap U = \varnothing$, this implies that $\forall\, \alpha \succ \beta, x_\alpha \notin U$. Therefore, x_α cannot be eventually in U. As a consequence, y can't be a limit point of $(x_\alpha)_{\alpha\in I}$.

- ⇐: We shall prove the contrapositive statement. Assume (X,τ) is not Hausdorff. Then there are two distinct points $x, y \in X$ with no disjoint neighborhoods. Let $\mathfrak{N}_x \equiv \{O \in \tau; x \in O\}$ and $\mathfrak{N}_y \equiv \{O \in \tau; y \in O\}$. We now these sets, when equipped with the reverse inclusion order, are directed systems (Proposition 116). Therefore, we might use Proposition 117 to consider $\mathfrak{N}_x \times \mathfrak{N}_y$ as a directed system, with the order given by $(\alpha,\beta) \prec (\alpha',\beta') \Leftrightarrow \alpha \prec_x \alpha'$ and $\beta \prec_y \beta'$.

  By hypothesis, we know that given $O \in \mathfrak{N}_x, U \in \mathfrak{N}_y$, it holds that $O \cap U \neq \varnothing$. Furthermore, since X is an open set, $\mathfrak{N}_x \times \mathfrak{N}_y$ is not empty. Therefore, the Axiom of Choice guarantees the existence of a net $\left(x_{(O,U)}\right)_{(O,U)\in\mathfrak{N}_x\times\mathfrak{N}_y}$ with $x_{(O,U)} \in O \cap U, \forall\, O \in \mathfrak{N}_x, U \in \mathfrak{N}_y$. Notice that such a sequence converges both to x and to y.

  Indeed, let O be a neighborhood of x. Given any neighborhood U of y, we know that $x_{(O,U)} \in O$. Furthermore, since $(O,U) \prec (O',U')$ if, and only if, $O' \subseteq O$ and $U' \subseteq U$, we have that whenever $(O,U) \prec (O',U')$, it holds that $x_{(O',U')} \in O' \cap U' \subseteq O \cap U \subseteq O$.

Thus, the net is eventually in O and it holds that x is a limit point for it. The same argument applies to y, and therefore $\left(x_{(O,U)}\right)_{(O,U)\in\mathfrak{N}_x\times\mathfrak{N}_y}$ admits more than one limit. ■

We might now give a wider notion of a subsequence* by defining a subnet.

Definition 125 [Subnet]:

Let (X,τ) be a topological space and let $(x_\alpha)_{\alpha\in I}$ be a net in X. We say a net $\left(y_\beta\right)_{\beta\in J}$ is a *subnet* of $(x_\alpha)_{\alpha\in I}$ whenever there is a function $\beta\mapsto\alpha_\beta$ respecting the following requirements:

i. $y_\beta = x_{\alpha_\beta}$;

ii. $\forall\,\alpha_0\in I, \exists\,\beta_0\in J; \alpha_\beta > \alpha_0, \forall\,\beta > \beta_0$. ♠

Remark:

Notice that we don't ask for the mapping $\beta\mapsto\alpha_\beta$ to be injective. If $\left(y_\beta\right)_{\beta\in J}$ is a subnet of $(x_\alpha)_{\alpha\in I}$, it might still hold that the cardinality of J is larger than the cardinality of I, for example. A subnet of a sequence is not necessarily a subsequence. In fact, it is possible for a sequence to have no convergent subsequences, but still have convergent subnets. ♣

Due to these complications, the definition of a subnet might seem useless or unnecessarily difficult, but it isn't. The following result guarantees that we are indeed working with an appropriate definition, for it generalizes a similar result found in metric spaces.

Theorem 126:

Let (X,τ) be a topological space and $(x_\alpha)_{\alpha\in I}$ be a net in such space. A point $x\in X$ is a cluster point of X if, and only if, $(x_\alpha)_{\alpha\in I}$ admits a subnet $\left(y_\beta\right)_{\beta\in J}$ with $y_\beta\to x$. □

Proof:

$\Leftarrow$: Suppose $(x_\alpha)_{\alpha\in I}$ admits a subnet $\left(y_\beta\right)_{\beta\in J}$ with $y_\beta\to x$. We want to prove that, if O is a neighborhood of x, $\forall\,\alpha\in I, \exists\,\beta>\alpha; x_\beta\in O$. Since $y_\beta\to x$, we know that $\exists\,\gamma\in J; y_\beta\in O, \forall\,\beta>\gamma$.

Let $\alpha_0\in I$. There is $\beta_0\in J$ such that $\alpha_\beta>\alpha_0, \forall\,\beta>\beta_0$. Let $\gamma_0\in J; y_\beta\in O, \forall\,\beta>\gamma_0$. We know there is $\delta\in J; \delta>\gamma_0$ and $\delta>\beta_0$. Since $\delta>\beta_0, \alpha_\delta>\alpha_0$. Since $\delta>\gamma_0, x_{\alpha_\delta}=y_\delta\in O$. Therefore, $\forall\,\alpha_0\in I, \exists\,\alpha_\delta\in I; x_{\alpha_\delta}\in O$. This proves that x is a cluster point of $(x_\alpha)_{\alpha\in I}$.

$\Rightarrow$: Let us assume x is a cluster point of $(x_\alpha)_{\alpha\in I}$. Then, for every neighborhood O of x, it holds that $\forall\,\alpha\in I, \exists\,\beta>\alpha; x_\beta\in O$.

Let $\mathfrak{N}=\{O\in\tau; x\in O\}$. Propositions 116 and 117 allow us to consider $\mathfrak{N}\times I$ as a directed system. Let us choose, $\forall\,(O,\gamma)\in\mathfrak{N}\times I$, $\alpha_{(O,\gamma)}\in I$ such that $\alpha_{(O,\gamma)}>\gamma$ and $x_{\alpha_{(O,\gamma)}}\in O$. It is possible to make this choice due to the fact that x is a cluster point of $(x_\alpha)_{\alpha\in I}$: $\forall\,\gamma\in I, \exists\,\beta>\gamma; x_\beta\in O$.

*Subsequences play an important role in the theory of Metric Spaces, but they are also not so interesting when dealing with Topology

Consider the net $\left(y_{(O,\gamma)}\right)_{(O,\gamma)\in\mathfrak{N}\times I}$ given by $y_{(O,\gamma)} = x_{\alpha_{(O,\gamma)}}$. Notice that $\forall\, \alpha_0 \in I$, one might pick any $O \in \mathfrak{N}$ and have $\alpha_{(O,\gamma)} \succ \alpha_0, \forall\, \gamma \succ \alpha_0$ (for $\alpha_{(O,\gamma)} \succ \gamma$ by definition). Therefore, $\left(y_{(O,\gamma)}\right)_{(O,\gamma)\in\mathfrak{N}\times I}$ is a subnet of $(x_\alpha)_{\alpha\in I}$.

We want to prove that $y_{(O,\gamma)} \to x$. Let $O \in \mathfrak{N}$ and $\gamma \in I$. If $(U, \delta) \succ (O, \gamma)$, then it holds that $U \subseteq O$ and $\delta \succ \gamma$. Therefore, $y_{(U,\delta)} = x_{\alpha_{(U,\delta)}} \in U \subseteq O$. ■

9 Compactness

Once again, we are going to impose extra conditions on topological spaces with the intention of obtaining a richer theory.

Definition 127 [Open Cover and Subcover]**:**

Let (X, τ) be a topological space. Let $\mathcal{A} \subset \tau$. We say $\mathcal{A}$ is an *open cover* of X if, and only if, $\bigcup_{O\in\mathcal{A}} O = X$. A family $\mathcal{S} \subseteq \mathcal{A}$ is said to be a *subcover* of $\mathcal{A}$ if it is also an open cover. ♠

Definition 128 [Compact Spaces and Subsets]**:**

Let (X, τ) be a topological space. X is said to be a *compact space* if, and only if, every open cover of X admits a finite subcover. We might also say that a subset $A \subseteq X$ is a *compact subspace* if it is a compact space when equipped with the relative topology. ♠

Proposition 129:

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f\colon X \to Y$ be a homeomorphism. X is compact if, and only if, Y is compact. □

Proof:

Suppose X is compact. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of Y. Then $\left\{f^{-1}(A_\lambda)\right\}_{\lambda\in\Lambda}$ is an open cover of X. Indeed,

$$\begin{aligned} Y &\subseteq \bigcup_{\lambda\in\Lambda} A_\lambda, \\ f^{-1}(Y) &\subseteq f^{-1}\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right), \\ X &\subseteq \bigcup_{\lambda\in\Lambda} f^{-1}\left(A_\lambda\right), \end{aligned} \tag{9.1}$$

where we used that $f^{-1}(Y) = X$ (for f is a bijection). Notice that the sets $f^{-1}\left(A_\lambda\right)$ are indeed open, for f is continuous.

Since X is compact, we know there is $n \in \mathbb{N}$ and $\lambda_k \in \Lambda, k \in \{k\}_{k=1}^{n}$ such that

$X \subseteq \bigcup_{k=1}^{n} f^{-1}(A_{\lambda_n})$. It follows then that

$$
\begin{aligned}
X &\subseteq \bigcup_{k=1}^{n} f^{-1}(A_{\lambda_n}), \\
f(X) &\subseteq f\left(\bigcup_{k=1}^{n} f^{-1}(A_{\lambda_n})\right), \\
Y &\subseteq \bigcup_{k=1}^{n} f\left(f^{-1}(A_{\lambda_n})\right), \\
Y &\subseteq \bigcup_{k=1}^{n} A_{\lambda_n},
\end{aligned} \tag{9.2}
$$

where the manipulations were possible due to the fact that f is bijective. Since we found a finite subcover of $\{A_\lambda\}_{\lambda\in\Lambda}$, Y is compact.

If we assumed Y was compact, the same argument with f^1 instead of f would prove X is compact. This concludes the proof. ■

The definition of compactness can also be cast in a different form.

Definition 130 [Finite Intersection Property]**:**

Let (X, τ) be a topological space. It is said to have the *finite intersection property* - also known as f.i.p. - if, and only if, given an arbitrary family of indices Λ and a family of closed sets $\{F_\lambda\}_{\lambda\in\Lambda}$ such that

$$
\bigcap_{k=1}^{n} F_{\lambda_k} \neq \varnothing, \tag{9.3}
$$

$\forall\, n \in \mathbb{N}$ and for any choice of $\lambda_k \in \Lambda, k \in \{i\}_{i=1}^{n}$, it holds that

$$
\bigcap_{\lambda\in\Lambda} F_\lambda \neq \varnothing, \tag{9.4}
$$

id est, if the intersections of finitely many elements of the family are non-empty, then the intersection of the whole family is also non-empty. ♠

Theorem 131:

A topological space is compact if, and only if, it has the finite intersection property. □

Proof:

$\Leftarrow$: Suppose (X, τ) is a topological space endowed with the finite intersection property. Let $A_{\lambda\,\lambda\in\Lambda}$ be an open cover of X. We want to prove it admits a finite subcover.

Notice that

$$
\begin{aligned}
\bigcup_{\lambda\in\Lambda} A_\lambda &= X, \\
\bigcap_{\lambda\in\Lambda} A_\lambda^c &= \varnothing.
\end{aligned} \tag{9.5}
$$

Since $\{A_\lambda^c\}_{\lambda\in\Lambda}$ is a family of closed sets with empty intersection and (X,τ) has the f.i.p., there has to be some $n \in \mathbb{N}$ and $\lambda_k \in \Lambda, k \in \{i\}_{i=1}^n$ such that

$$\bigcap_{k=1}^{n} A_{\lambda_k}^c = \varnothing. \tag{9.6}$$

Otherwise, the finite intersection property would imply the intersection of the whole family is non-empty, which is absurd.

Notice now that

$$\begin{aligned}\bigcap_{k=1}^{n} A_{\lambda_k}^c &= \varnothing,\\ \bigcup_{k=1}^{n} A_{\lambda_k} &= X,\end{aligned} \tag{9.7}$$

and thus we have found a finite subcover of $\{A_\lambda\}_{\lambda\in\Lambda}$.

$\Rightarrow$: Suppose now that (X,τ) is a compact space. The contrapositive of the finite intersection property can be proven by essentially reversing the steps taken to prove that the f.i.p. implies compactness.

■

Proposition 132:

Let (X,τ) be a compact space. If $F \subseteq X$ is closed, it is compact. □

Proof:

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets in F such that $\bigcap_{\lambda\in\Lambda} A_\lambda = \varnothing$. Due to Corollary 28, we know $\{A_\lambda\}_{\lambda\in\Lambda}$ is a collection of closed sets with respect to the topology on X as well. Since (X,τ) is compact, Theorem 131 implies it has the f.i.p. Therefore, we know there must exist $n \in \mathbb{N}, \lambda_k \in \Lambda, k \in \{i\}_{i=1}^n$ such that $\bigcap_{k=1}^n A_{\lambda_k} = \varnothing$. Thus, F also respects the f.i.p., and Theorem 131 guarantees its compactness. ■

Lemma 133:

Let (X,τ) be a Hausdorff space. If $K \subseteq X$ is compact and $x \notin K$, then there are disjoint open sets O and U such that $K \subseteq U, x \in O$. □

Proof:

Let $y \in K$. The Hausdorff property guarantees the existence of disjoint open sets O_y and U_y such that $y \in U_y, x \in O_y$. Notice that the collection $\{U_y\}_{y\in K}$ defined in this manner is an open cover of K. Thus, since K is compact, there is $n \in \mathbb{N}, y_k \in K, k \in \{i\}_{i=1}^n$ such that $\{U_{y_k}\}_{k=1}^n$ is an open cover of K. Since arbitrary unions of open sets are open sets, $U = \bigcup_{k=1}^n U_{y_k}$ is an open set containing K.

Notice now that the set $O = \bigcap_{k=1}^n O_{y_k}$ is an open set - for it is a finite intersection of open sets - and it contains x - for $x \in O_y, \forall\, y \in K$.

Finally, notice now that $O \cap U = \varnothing$. Since $U_y \cap O_y = \varnothing, \forall\, y \in K$, and $O \subseteq O_{y_k}, \forall\, k \in \{i\}_{i=1}^n$, it holds that $O \cap U_{y_k} = \varnothing, \forall\, k \in \{i\}_{i=1}^n$. Since $U = \bigcup_{k=1}^n U_{y_k}$, we conclude $U \cap O = \varnothing$, and thus we have proven the claim. ■

Proposition 134:

Let (X, τ) be a Hausdorff space. If $K \subseteq X$ is compact, it is closed. □

Proof:

Given $x \in K^c$, Lemma 133 guarantees the existence of an open set O_x such that $x \in O_x \subseteq K^c$. Notice then that $K^c = \bigcup_{x \in K^c} O_x$ and, being a union of open sets, we conclude that K^c is open. Thus, K is closed. ■

Theorem 135:

Let (X, τ) be a compact and Hausdorff space. Then (X, τ) is normal. □

Proof:

Let $A, B \subseteq X$ be closed sets. Since X is compact, Proposition 132 guarantees A and B are compact. Pick $x \in B$. Since X is Hausdorff, Lemma 133 guarantees there are disjoint open sets O_x and U_x such that $A \subseteq U_x$ and $x \in O_x$. Notice that $\{O_x\}_{x \in B}$ is an open cover of B, which is compact. Thus, we may conclude there are $n \in \mathbb{N}, x_k \in B, k \in \{i\}_{i=1}^{n}$ such that $\{O_{x_k}\}_{k=1}^{n}$ covers B.

If we take $O = \bigcup_{k=1}^{n} O_{x_k}$ and $U = \bigcap_{k=1}^{n} U_{x_k}$, we find that these are both open sets - for they are finite unions or intersections of open sets -, $A \subseteq U$ - for $A \subseteq U_x, \forall\, x \in B$ - and $B \subseteq O$ - $\{O_{x_k}\}_{k=1}^{n}$ covers B.

Finally, $O \cap U = \varnothing$. Since $U_x \cap O_x = \varnothing, \forall\, x \in B$, and $U \subseteq U_{x_k}, \forall\, k \in \{i\}_{i=1}^{n}$, it holds that $U \cap O_{x_k} = \varnothing, \forall\, k \in \{i\}_{i=1}^{n}$. Since $O = \bigcup_{k=1}^{n} O_{x_k}$, we conclude $U \cap O = \varnothing$.

Thus, we have proven that given any two closed sets $A, B \subseteq X$, there are $O, U \in \tau$ such that $A \subseteq U, B \subseteq O, O \cap U = \varnothing$, *id est*, (X, τ) is normal. ■

Corollary 136:

Let (X, τ) be a compact Hausdorff space and let $A, B \subseteq X$ be closed sets. Let $a, b \in \mathbb{R}, a < b$. Then there is some continuous function $f\colon A \to B$ with $f(A) = \{a\}$ and $f(B) = \{b\}$. □

Proof:

Theorem 135 guarantees (X, τ) is normal. Thus, we might apply Urysohn's Lemma and the result is proven. ■

Definition 137 [Weakly Sequentially Compact]**:**

Let (X, τ) be a topological space. If every sequence in X has a cluster point, (X, τ) is said to be *weakly sequentially compact*. ♠

Definition 138 [Sequentially Compact]**:**

Let (X, τ) be a topological space. If every sequence in X has a convergent subsequence, (X, τ) is said to be *sequentially compact*. ♠

Lemma 139:

Let (M, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence defined on M. x is a cluster point of $(x_n)_{n \in \mathbb{N}}$ if, and only if, $(x_n)_{n \in \mathbb{N}}$ has a subsequence converging to x. □

Proof:

Assuming x is a cluster point of $(x_n)_{n \in \mathbb{N}}$, we know that given any neighborhood $\mathcal{B}$ of x, there are infinitely many $m \in \mathbb{N}$ such that $x_m \in \mathcal{B}$.

Let us consider the sets $\mathcal{B}_{\frac{1}{n}}(x)$, which are all neighborhoods of x. Thus, there are infinitely many elements of the sequence in each one of them. $\forall\, n \in \mathbb{N}$, let us define

$$m_n \equiv \min\left\{p \in \mathbb{N}; x_p \in \mathcal{B}_{\frac{1}{n}}(x)\right\}. \tag{9.8}$$

Notice that y_{m_n} defines a subsequence of x_n. Furthermore, given $n \in \mathbb{N}$, it holds that

$$y_{m_n} \in \mathcal{B}_{\frac{1}{n}}(x) \subseteq \mathcal{B}_{\frac{1}{p}}(x), \forall\, p > n. \tag{9.9}$$

Thus, y_{m_n} is eventually in any $\mathcal{B}_{\frac{1}{n}}(x)$. In Proposition 94 we have proved this sets are a neighborhood basis for the metric topology at x. Thus, given a neighborhood O of x, there is some $n \in \mathbb{N}$ such that $\mathcal{B}_{\frac{1}{n}}(x) \subseteq O$. Since y_{m_n} is eventually in $\mathcal{B}_{\frac{1}{n}}(x)$, it is eventually in O. Since the subsequence is eventually in any neighborhood of x, we conclude $y_{m_n} \to x$, as desired.

Suppose now that $(x_n)_{n\in\mathbb{N}}$ has a subsequence converging to x. This means the subsequence is eventually in every neighborhood of x. Since the subsequence has infinitely many terms, this can only happen in the sequence is frequently in every neighborhood of x. ■

Proposition 140:

A metric space is weakly sequentially compact if, and only if, it is sequentially compact. □

Proof:

Lemma 139. ■

Theorem 141:

Let (X, τ) be a compact topological space. (X, τ) is weakly sequentially compact. □

Proof:

Suppose (X, τ) was not weakly sequentially compact, *id est*, suppose there is some sequence $(x_n)_{n\in\mathbb{N}}$ with no cluster points, *id est*, a sequence such that

$$\forall\, x \in X, \exists\, O_x \in \tau, \exists\, n \in \mathbb{N}; x \in O_x, x_m \notin O_x, \forall\, m \geqslant n. \tag{9.10}$$

Thus, there are finitely many elements of the sequence in each O_x (otherwise, the sequence would be frequently in some O_x, which is forbidden by hypothesis).

Notice that $\{O_x\}_{x\in X}$ is an open cover of X. Sine X is compact, it admits a finite subcover, and thus $X \subseteq \bigcup_{k=1}^{n} O_{x_k}$ for some $n \in \mathbb{N}, x_k \in X, k \in \{i\}_{i=1}^{n}$. Since there are finitely many elements of x_n in each O_x and $\bigcup_{k=1}^{n} O_{x_k}$ is the reunion of finitely many O_x, there are finitely many elements of x_n in $\bigcup_{k=1}^{n} O_{x_k}$. However, we know there are infinitely many elements of x_n in X, and therefore we have arrived at a contradiction. This forces to conclude (X, τ) is indeed weakly sequentially compact. ■

Definition 142 [Totally Bounded Metric Space]**:**

Let (M, d) be a metric space. (M, d) is said to be *totally bounded* if, and only if, $\forall\, \epsilon > 0, \exists\, n \in \mathbb{N}, x_k \in M, k \in \{i\}_{i=1}^{n}$ such that $M \subseteq \bigcup_{k=1}^{n} \mathcal{B}_\epsilon(x_k)$. ♠

Proposition 143:

Let (M, d) be a totally bounded metric space. Then it is separable. □

Proof:

Given $n \in \mathbb{N}$, we know there are $m_n \in \mathbb{N}, x_k^{(n)} \in X, k \in \{i\}_{i=1}^{m_n}; M \subseteq \bigcup_{k=1}^{m_n} \mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$. Notice that the set $A = \left\{x_k^{(n)}; k \in \{i\}_{i=1}^{m_n}, n \in \mathbb{N}\right\}$ is countable, for it is the countable union of finite sets. A is dense in M.

Indeed, let $x \in M, \epsilon > 0$. Due to the Archimedean property of the real numbers, we know there is some $n \in \mathbb{N}; \frac{1}{n} < \epsilon$. Since $M \subseteq \bigcup_{k=1}^{m_n} \mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$, we know there is some $k \in \{i\}_{i=1}^{m_n}$ such that $x \in \mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$. Thus, $\exists\, p \in A; x \in \mathcal{B}_{\frac{1}{n}}(p) \subseteq \mathcal{B}_\epsilon(p)$. Therefore, A is dense in M in the sense of metric spaces and Theorem 102 implies $\overline{A} = M$. As A is countable, M is separable. ■

Proposition 144:

Let (M, d) be a complete and totally bounded metric space. Then (M, d) is weakly sequentially compact as well. □

Proof:

Let $(x_l)_{l\in\mathbb{N}}$ be a sequence of points of M. Given $n \in \mathbb{N}$, we know there are $m_n \in \mathbb{N}, x_k^{(n)} \in X, k \in \{i\}_{i=1}^{m_n}; M \subseteq \bigcup_{k=1}^{m_n} \mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$.

Let $n = 1$. We know that $\forall\, l \in \mathbb{N}, \exists\, k \in \{k\}_{k=1}^{m_1}; x_l \in \mathcal{B}_1\left(x_k^{(1)}\right)$. Since the sequence $(x_l)_{l\in\mathbb{N}}$ has infinitely many terms and there are only finitely many $x_k^{(1)}$, there is at least one $k_1 \in \{k\}_{k=1}^{m_1}$ such that there are infinitely many terms of $(x_l)_{l\in\mathbb{N}}$ laying on $\mathcal{B}_1\left(x_{k_1}^{(1)}\right)$. These terms define a subsequence $y_p^{(1)}$.

Now, for each $n \in \mathbb{N}$, we can iterate this process. Given the sequence $(y_l^{(n)})_{l\in\mathbb{N}}$, we know that $\forall\, l \in \mathbb{N}, \exists\, k \in \{k\}_{k=1}^{m_{n+1}}; y_l^{(n)} \in \mathcal{B}_{\frac{1}{n+1}}(x_k) \cap \mathcal{B}_{\frac{1}{n}}(x_{k_n})$, since $y_l^{(n)} \in \mathcal{B}_{\frac{1}{n}}(x_{k_n}), \forall\, l \in \mathbb{N}$ an M is totally bounded. There is at least one $k_n \in \{k\}_{k=1}^{m_{n+1}}$ such that infinitely many terms of $(y_l^{(n)})_{l\in\mathbb{N}}$ lay on $\mathcal{B}_{\frac{1}{n+1}}(x_{k_n})$.

The Axiom of Choice now allows us to, $\forall\, n \in \mathbb{N}$, pick $l_n \in \mathbb{N}$ in order to form a subsequence $(x_{l_n})_{n\in\mathbb{N}}$ such that $x_{l_m} \in \mathcal{B}_{\frac{1}{n}}(x_{k_n}), \forall\, m > n$. x_{l_1} is an element of $(y_l^{(1)})_{l\in\mathbb{N}}$, x_{l_2} is an element of $(y_l^{(2)})_{l\in\mathbb{N}}$ such that $l_2 > l_1$ (which is possible due to the fact that there are infinitely many terms on $(y_l^{(2)})_{l\in\mathbb{N}}$) and so on.

$(x_{l_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed, let $\epsilon > 0$. The Archimedean property of the real numbers guarantees the existence of $m \in \mathbb{N}; \frac{2}{m} < \epsilon$. By construction, $x_{l_p}, x_{l_q} \in \mathcal{B}_{\frac{1}{m}}(x_{k_m}), \forall\, p, q > m$. Therefore, $d(x_{l_p}, x_{l_q}) < \frac{2}{m} < \epsilon, \forall\, p, q > m$.

Since $x \in M$ is a limit point of a subsequence of $(x_n)_{n\in\mathbb{N}}$, Lemma 139 guarantees x is a cluster point of $(x_n)_{n\in\mathbb{N}}$. ■

Lemma 145:

Let (M, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. If a subsequence $(x_{n_m})_{m\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in M$, then $x_n \to x$ as well. □

Proof:

Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, we know that

$$\forall\, \epsilon > 0, \exists\, n_\epsilon \in \mathbb{N}; d(x_m, x_n) < \epsilon, \forall\, m, n > n_\epsilon. \tag{9.11}$$

Given $\epsilon > 0$, let m_ϵ be such that $n_{m_\epsilon} \geqslant n_\epsilon$. There is such an m_ϵ due to the fact that the subsequence $(x_{n_m})_{m\in\mathbb{N}}$ has infinitely many terms. We have that

$$\forall\, \epsilon > 0, \exists\, n_\epsilon \in \mathbb{N}; d(x_{n_m}, x_n) < \epsilon, \forall\, n > n_\epsilon, \forall\, m > m_\epsilon. \tag{9.12}$$

Since $x_{n_m} \to x$, we know that

$$\forall\, \epsilon > 0, \exists\, p_\epsilon \in \mathbb{N}; d(x_{n_m}, x) < \epsilon, \forall\, m > p_\epsilon. \tag{9.13}$$

Keeping this in mind, let $\epsilon > 0$. Then we know that

$$\begin{aligned} d(x_{n_m}, x) &< \epsilon, \forall\, m > \max p_\epsilon, m_\epsilon \equiv q_\epsilon, \\ d(x_{n_m}, x_n) &< \epsilon, \forall\, m > q_\epsilon, \forall\, n > n_{q_\epsilon}. \end{aligned} \tag{9.14}$$

Due to the triangle inequality, we see that $\forall\, m > q_\epsilon, \forall\, n > n_{q_\epsilon}$,

$$d(x_n, x) < d(x_{n_m}, x_n) + d(x_{n_m}, x) < 2\epsilon. \tag{9.15}$$

As a consequence, we conclude that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; d(x_n, x) < \epsilon, \forall\, n > n_0, \tag{9.16}$$

id est, $x_n \to x$, as desired. ■

Theorem 146:

Let (M, d) be a metric space. The following statements are equivalent:

i. *it is compact;*

ii. *it is weakly sequentially compact;*

iii. *it is sequentially compact;*

iv. *it is complete and totally bounded.* □

Proof:

i. ⇒ ii. Theorem 141;

ii. ⇔ iii. Proposition 140;

iii. ⇒ iv. Suppose (M, d) is sequentially compact, *id est*, that any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of M has a convergent subsequence. Due to Lemma 145, this implies (M, d) is complete.

Suppose (M, d) is not totally bounded, *id est*,

$$\exists\, \epsilon > 0; \forall\, n \in \mathbb{N}, \forall\, x_k \in M, k \in \{k\}_{k=1}^{n}, \exists\, x \in M; x \notin \bigcup_{k=1}^{n} \mathcal{B}_\epsilon(x_k). \tag{9.17}$$

Let $x_1 \in M$. For each $n \in \mathbb{N}^*$, let us pick $x_{n+1} \in \left(\bigcup_{k=1}^{n} \mathcal{B}_\epsilon(x_k)\right)^c$. Notice that, by construction, $\forall\, n \in \mathbb{N}, d(x_n, x_m) \geqslant \epsilon, \forall\, m < n$. We may rewrite this as $d(x_n, x_m) \geqslant \epsilon\, \forall\, n \neq m$. As a consequence, $(x_n)_{n\in\mathbb{N}}$ has no cluster points. Indeed, suppose $x_n \in \mathcal{B}_{\frac{\epsilon}{3}}(x)$ for some $n \in \mathbb{N}$. Then $d(x, x_n) \leqslant \frac{\epsilon}{3}$. We already know that $\epsilon \leqslant d(x_n, x_m)\, \forall\, m \neq n$. The triangle inequality yields, $\forall\, m \neq n$,

$$
\begin{aligned}
\epsilon \leqslant d(x_n, x_m) &\leqslant d(x_n, x) + d(x, x_m), \\
\epsilon &< \frac{\epsilon}{3} + d(x, x_m), \\
\frac{2\epsilon}{3} &< d(x, x_m).
\end{aligned} \tag{9.18}
$$

Therefore, $x_m \notin \mathcal{B}_{\frac{\epsilon}{3}}(x)$. Since the argument holds for every $x \in M$, there are no points in M such that $(x_n)_{n\in\mathbb{N}}$ is frequently within every neighborhood. Therefore, $(x_n)_{n\in\mathbb{N}}$ has no cluster points, which implies, through Lemma 139, that $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Therefore, (M, d) is not sequentially compact. This means we have proved the contrapositive to the statement we wanted to prove, which means that, indeed, (M, d) being sequentially compact implies (M, d) being totally bounded.

iv. $\Rightarrow$ i. Proposition 143 guarantees a totally bounded metric space is separable. Proposition 103 guarantees a separable metric space is a Lindelöf space. Thus, every open cover of (M, d) has a countable subcover. As a consequence, we just need to prove that every countable open cover of (M, d) admits a finite subcover.

Let $\{A_n\}_{n\in\mathbb{N}}$ be an open cover of (M, d). Suppose, by contradiction, M is not compact, *id est*, $\forall\, n \in \mathbb{N}, M \neq \bigcup_{k=1}^{n} A_n$. We may then define $B_n \equiv X \backslash \bigcup_{k=1}^{n} A_n$. Notice that $B_{n+1} \subseteq B_n, \forall\, n \in \mathbb{N}$. Since $B_n \neq \varnothing, \forall\, n \in \mathbb{N}$, we may define a sequence by choosing $x_n \in B_n, \forall\, n \in \mathbb{N}$. Proposition 144 guarantees $(x_n)_{n\in\mathbb{N}}$ has a cluster point. However, Propositions 94 and 101 guarantee $x \in B_n, \forall\, n \in \mathbb{N}$, since B_n is always closed (for it is the complement of a union of open sets, *id est*, the complement of an open set) and removing finitely many terms of $(x_n)_{n\in\mathbb{N}}$ doesn't change the fact that $(x_n)_{n\in\mathbb{N}}$ is frequently in any neighborhood of x.

Since $x \in B_n, \forall\, n \in \mathbb{N}$, we get

$$
\begin{aligned}
x \in \bigcap_{n=1}^{+\infty} B_n &= \bigcap_{n=1}^{+\infty} \left[\bigcup_{k=1}^{n} A_k\right]^c, \\
&= \left[\bigcup_{k=1}^{+\infty} A_k\right]^c, \\
&= M^c, \\
&= \varnothing.
\end{aligned} \tag{9.19}
$$

This is a contradiction, and thus the hypothesis that M is not compact is false. Hence, the proof is complete. ■

Finally, we may give a complete description of all compact sets in $\mathbb{R}^n$ by means of the Heine-Borel Theorem.

Theorem 147 [Heine-Borel]**:**

Consider the metric space $(\mathbb{R}^n, d)$*, where* d *is the standard Euclidean metric. A subset* $K \subseteq \mathbb{R}^n$ *is compact if, and only if, it is closed and bounded.* □

Proof:

- $\Rightarrow$: Assume K is compact. Then Lemma 89 and Proposition 134 guarantee K is closed. Theorem 146 guarantees K is totally bounded. Thus, $\exists\, n \in \mathbb{N}, x_k \in K, k \in \{k\}_{k=1}^{n}\, ; K \subseteq \bigcup_{k=1}^{n} \mathcal{B}_1(x_k)$.

  If we define $m \equiv \max_{1 \leqslant k \leqslant n} \{d(x_k, 0)\}$, then it holds that $U \subseteq \mathcal{B}_{1+m}(0)$. Indeed, suppose $x \in U$. Then $x \in \bigcup_{k=1}^{n} \mathcal{B}_1(x_k)$, which means $x \in \mathcal{B}_1(x_k)$ for some $k \in \{k\}_{k=1}^{n}$. Thus, we know that $d(x, x_k) < 1$ and $d(x_k, 0) \leqslant m$. The triangle inequality yields

$$\begin{aligned} d(x,0) &\leqslant d(x,x_k) + d(x_k,0), \\ &< 1 + m, \end{aligned} \tag{9.20}$$

  as claimed. This proves K is bounded.

- $\Leftarrow$: Assume now K is closed and bounded. Since $(\mathbb{R}^n, d)$ is complete and K is closed, K is complete as well*.

  Since K is bounded, we know there is some $r > 0$ such that $K \subseteq \mathcal{B}_r(0)$. As a consequence, we see that $K \subseteq (-r, r)^n$, the hypercube with side $2r$ centered at the origin.

  Given $\epsilon > 0$, let $\delta < \frac{2\epsilon}{\sqrt{n}}$. Notice that a hypercube of side δ is always contained in an open ball of radius ϵ. Indeed, the hypercube's diagonal is given by $D = \sqrt{n\delta^2} = \sqrt{n}\delta < 2\epsilon$, which is the diameter of an open ball of radius ϵ. Thus, if we cover K with finitely many cubes of side δ, we may also cover it with finitely many open balls of radius ϵ, and that proves K is totally bounded.

  The interval $(-r, r)$ can be covered by $\lceil \frac{2r}{\delta} \rceil$ intervals of size δ, where $\lceil x \rceil$ denotes the smallest integer larger than x. Similarly, the hypercube $(-r, r)^n$ can be covered by $\lceil \frac{2r}{\delta} \rceil^n$ hypercubes of side δ. Since $K \subseteq (-r, r)^n$, this means K can be covered by finitely many hypercubes of side δ, or, equivalently, by finitely many open balls of radius ϵ. Hence, K is totally bounded.

  Since K is complete and totally bounded, Theorem 146 ensures K is compact, concluding the proof.

■

Theorem 148 [Bolzano-Weierstrass]**:**

Consider the metric space $(\mathbb{R}^n, d)$*, where* d *is the Euclidean metric. If a sequence* $(x_n)_{n \in \mathbb{N}}$ *of elements of* $\mathbb{R}^n$ *is bounded,* id est, *if* $\exists\, r > 0; x_n \in \mathcal{B}_r(0), \forall\, n \in \mathbb{N}$*, then* $(x_n)_{n \in \mathbb{N}}$ *has a convergent subsequence.* □

*This can be proven easily by employing Propositions 94 and 101.

Proof:

Since $x_n \in \mathcal{B}_r(0), \forall\, n \in \mathbb{N}$, it holds that $x_n \in \overline{\mathcal{B}_r(0)}, \forall\, n \in \mathbb{N}$. Notice that $\overline{\mathcal{B}_r(0)} = \{x \in \mathbb{R}^n; d(x,0) \leqslant r\}$. Since $\overline{\mathcal{B}_r(0)}$ is closed and bounded (for $\overline{\mathcal{B}_r(0)} \subseteq \mathcal{B}_{r+1}(0)$), the Heine-Borel Theorem ensures it is compact. Since it is compact, Theorem 146 ensures it is sequentially compact. Hence, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. ■

Proposition 149:

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Suppose (X, τ_X) is compact. Let $f\colon X \to Y$ be a continuous function. Then $\operatorname{Ran} f = f(X)$ is compact. □

Proof:

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a open cover of $\operatorname{Ran} f$ in the relative topology. We know then there are $O_\lambda \in \tau_Y$ such that $U_\lambda = O_\lambda \cap \operatorname{Ran} f, \forall\, \lambda \in \Lambda$. We see that

$$\begin{aligned} \operatorname{Ran} f &\subseteq \bigcup_{\lambda\in\Lambda} O_\lambda, \\ f^{-1}(\operatorname{Ran} f) &\subseteq f^{-1}\left(\bigcup_{\lambda\in\Lambda} O_\lambda\right), \\ X &\subseteq \bigcup_{\lambda\in\Lambda} f^{-1}(O_\lambda). \end{aligned} \tag{9.21}$$

Since f is continuous, the sets $f^{-1}(O_\lambda)$ are open. Since X is compact, there is a finite set I such that

$$\begin{aligned} X &\subseteq \bigcup_{i\in I} f^{-1}(O_{\lambda_i}), \\ f(X) &\subseteq f\left(\bigcup_{i\in I} f^{-1}(O_{\lambda_i})\right), \\ \operatorname{Ran} f &\subseteq \bigcup_{i\in I} O_{\lambda_i}, \\ \operatorname{Ran} f &\subseteq \bigcup_{i\in I} O_{\lambda_i} \cap \operatorname{Ran} f, \\ \operatorname{Ran} f &\subseteq \bigcup_{i\in I} U_{\lambda_i}, \end{aligned} \tag{9.22}$$

proving $\{U_\lambda\}_{\lambda\in\Lambda}$ admits a finite subcover, and hence that $\operatorname{Ran} f$ is compact. ■

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