A Brief Introduction to Classical Field Theory

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(Dated: February 15, 2020)

In this work, I present an introduction to classical field theory by exploring the limiting case of N coupled harmonic oscillators as $N \to \infty$ in order to obtain the equation of motion for a vibrating string. The Euler-Lagrange Equations for a collection of N three-dimensional fields are presented and, finally, Noether's Theorem is proved, with the stress-energy tensor and the conservation of electric charge due to gauge invariance in QED being given as examples of application.

Keywords: classical field theory, Noether's theorem

I. VIBRATING STRINGS

Suppose you want to describe the motion of a vibrating string of length L and linear mass density μ with clamped ends. A simple model for this situation is considering a series of coupled harmonic oscillators confined to movement in a single direction.Suppose you want to describe the motion of a vibrating string of length L and linear mass density μ with clamped ends. A simple model for this situation is considering a series of coupled springs, with spring constant k, confined to movement in a single direction. Such oscillators should each have mass $m = \mu a$, where $a = \frac{L}{N+1}$ is the distance between two consecutive oscillators. The height of the oscillator located at the position x_i is given by a function $q_i(t)$. As the string should be clamped at its ends, we impose that $q_0(t) = q_{N+1}(t) = 0$ at all times.

The kinetic energy K_i stored in the i-th oscillator is

$$K_i = \frac{m}{2}\dot{q}_i^2. \tag{1}$$

On the other hand, the potential energy U_i stored in the spring between positions x_i and x_{i+1} is given by

$$U_i = \frac{k}{2}(l_i - l)^2, \tag{2}$$

where l stands for the rest length of the spring (which is equal for every spring here considered) and l_i is the length of the spring between positions x_i and x_{i+1}, which can be obtained through trigonometry. If the angle the spring makes with the horizontal line is θ, then we know that $l_i = a \cdot \sec\theta$. Furthermore, we know that $\tan\theta = \frac{q_{i+1}-q_i}{a}$. Thus, as $\tan^2\theta + 1 = \sec^2\theta$, it follows that

$$l_i = a\sqrt{1 + \left(\frac{q_{i+1} - q_i}{a}\right)^2}. \tag{3}$$

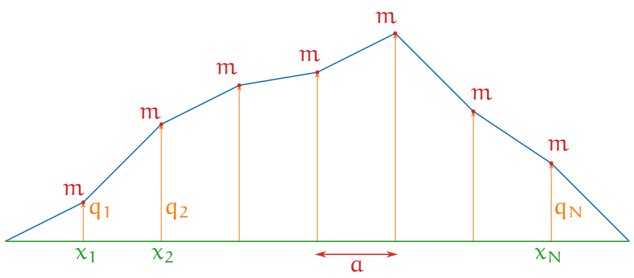

Figure 1. A simple model for a vibrating string of length L and linear mass density μ consists of N coupled harmonic oscillators, each a distance $a = \frac{L}{N}$ apart from its neighbors with mass $m = \mu a$

We still must discover what is the value of l. We know that $L_0 = (N+1)l$ is the rest length of the string, while it's current length is (roughly[1]) $L = (N+1)a$. If a tension τ is being applied in the string, then we know from Hooke's Law that $\tau = -\frac{k}{N+1}\left((N+1)l - (N+1)a\right)$ (the factor $\frac{k}{N+1}$ comes from associating springs in series) and it follows that

$$\begin{aligned} \frac{\tau}{k} &= a - l, \\ l &= a - \frac{\tau}{k}. \end{aligned} \tag{4}$$

Thus, we have that the energy stored in each spring

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[1] Don't bother with this approximation right now, we are going to justify it in a minute or two.

is given by

$$U_i = \frac{k}{2}\left(a\sqrt{1+\left(\frac{q_{i+1}-q_i}{a}\right)^2}+\frac{\tau}{k}-a\right)^2 ,$$
$$= \frac{k}{2}a^2\left(1+\left(\frac{q_{i+1}-q_i}{a}\right)^2\right)+\frac{k}{2}\left(a-\frac{\tau}{k}\right)^2$$
$$- ka\sqrt{1+\left(\frac{q_{i+1}-q_i}{a}\right)^2}\left(a-\frac{\tau}{k}\right). \quad (5)$$

We will now make the assumption that the string is vibrating under a regime of small oscillations, *i.e.*, $|q_{i+1}-q_i| \ll a$. Without this approximation, even though the springs are being treated as linear, we would obtain a nonlinear behaviour, which is more appropriate for a deeper analysis of the problem. For simplicity, we are going to assume the string is being held really tight and we can only disturb it slightly.

This assumption justifies the approximation we made earlier that the length of the string is $L = (N+1)a$ (which is, in fact, also the definition of a we provided). If the disturbances are small, indeed the length of string is very close to $L = (N+1)a$ and our treatment is justified.

By Taylor expanding the square root in Eq. 5, it follows that

$$U_i = \frac{k}{2}a^2\left(1+\left(\frac{q_{i+1}-q_i}{a}\right)^2\right)+\frac{k}{2}\left(a-\frac{\tau}{k}\right)^2$$
$$- ka\left(1+\frac{1}{2}\left(\frac{q_{i+1}-q_i}{a}\right)^2\right)\left(a-\frac{\tau}{k}\right),$$
$$= \frac{k}{2}a^2+\frac{k}{2}\left(q_{i+1}-q_i\right)^2+\frac{k}{2}\left(a-\frac{\tau}{k}\right)^2+a\tau$$
$$- ka^2+\frac{\tau}{2a}\left(q_{i+1}-q_i\right)^2-\frac{k}{2}\left(q_{i+1}-q_i\right)^2,$$
$$= -\frac{k}{2}a^2+\frac{k}{2}\left(a-\frac{\tau}{k}\right)^2+a\tau+\frac{\tau}{2a}\left(q_{i+1}-q_i\right)^2. \quad (6)$$

We might now calculate the Lagrangian $\mathfrak{L} = K - U$ by summing Eqs. (1) and (6) over every mass and spring. Recalling that constant terms may be dropped, for they leave the Euler-Lagrange Equations unaltered, it follows that

$$\mathfrak{L} = \sum_{i=1}^{N} K_i - \sum_{i=0}^{N} U_i,$$
$$= \sum_{i=1}^{N}\frac{m}{2}\dot{q}_i^2 - \sum_{i=0}^{N}\frac{\tau}{2a}\left(q_{i+1}-q_i\right)^2. \quad (7)$$

So far, we have only dealt with a discrete approximation for the string problem. Let us try to figure out a way to take the limit $N \to \infty$ properly and obtain a full description of the string.

The first step is simple: we are currently using a finite number of degrees of freedom through the generalized coordinates q_i. If we want to describe a continuous string, we should drop this description in favor of something that accepts a continuous indexation. Thus, we are going to define a *field* $\phi(x,t)$ such that $\phi(x_i,t) = q_i(t), \forall\, i \in \{i\}_{i=0}^{N+1}, \forall\, t$. Furthermore, we are going to write $\Delta x \equiv a$ from now on, since this is the spatial variation between two oscillators and our plan is to take such a value to zero in the limiting case.

Recalling that $m = \mu a$, our Lagrangian should now be written as

$$\mathfrak{L} = \sum \Delta x\left[\frac{\mu}{2}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2\right]$$
$$- \sum \Delta x\left[\frac{\tau}{2}\left(\frac{\phi(x+\Delta x,t)-\phi(x,t)}{\Delta x}\right)^2\right]. \quad (8)$$

By taking the limit as $N \to \infty$, and recalling that $L = N\Delta x$ remains constant, we obtain

$$\mathfrak{L} = \int \frac{\mu}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 dx\,. \quad (9)$$

This motivates us to define a new function, named *Lagrangian density* and denoted by $\mathcal{L}$, such that

$$\mathfrak{L} = \int \mathcal{L}\, dx\,. \quad (10)$$

II. EULER-LAGRANGE EQUATIONS

If we want the description in terms of the Lagrangian density to be useful, we must obtain the equations of motion described by this quantity. The procedure is fairly similar to the way one obtains the Euler-Lagrange Equations for usual Lagrangians: apply Hamilton's Principle considering every possible field under the given boundary conditions.

As the action is defined as $S = \int_{t_1}^{t_2}\mathfrak{L}\,dt$, in terms of the Lagrangian density it becomes

$$S = \int_{t_1}^{t_2}\int_{x_1}^{x^2}\mathcal{L}(\phi,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial t},x,t)\,dx\,dt\,. \quad (11)$$

In a more general way, let us consider a Lagrangian density describing N fields, each of them denoted by

ϕ_i and collectively described as ϕ, in three spatial dimensions and one time dimension. The action is then written as

$$S = \int_\Omega \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial t}, \boldsymbol{\nabla}\phi, \mathbf{x}, t\right) d^4x\,. \tag{12}$$

Hamilton's Principle states that $\delta S = 0$. From this knowledge, we will be able to obtain the Euler-Lagrange Equations for our field theory.

With implicit summation over repeated indexes, we have

$$\delta S = \int_{t_1}^{t_2}\int_{\mathcal{V}} \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\,\delta\phi_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\,\delta\dot{\phi}_\alpha + \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\,\delta\boldsymbol{\nabla}\phi_\alpha\, d^3x\, dt = 0. \tag{13}$$

Let us first deal with the second term (the first term is already as simple as we want). Notice that

$$\begin{aligned}\int_\Omega \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\,\delta\dot{\phi}_\alpha\, d^4x &= \int_\Omega \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\frac{\partial\,\delta\phi_\alpha}{\partial t}\, d^4x\,,\\ &= \int_{\mathcal{V}} \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\,\delta\phi_\alpha\right|_{t_1}^{t_2} d^3x\\ &\quad - \int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right)\delta\phi_\alpha\, d^4x\,. \end{aligned} \tag{14}$$

As said earlier, we are considering every possible field *under the given boundary conditions*. Thus, $\left[\delta\phi_\alpha\right|_{t_1}^{t_2} = 0$. Thus,

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\,\delta\dot{\phi}_\alpha\, d^4x = -\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right)\delta\phi_\alpha\, d^4x\,. \tag{15}$$

We must then apply a similar argument to the third term in Eq. (13).

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\,\delta\boldsymbol{\nabla}\phi_\alpha\, d^4x = \int_\Omega \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\boldsymbol{\nabla}(\delta\phi_\alpha)\, d^4x\,. \tag{16}$$

If we recall that

$$\boldsymbol{\nabla}\cdot(f\mathbf{v}) = f\boldsymbol{\nabla}\cdot\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}f, \tag{17}$$

then it follows from Gauss's Theorem that

$$\begin{aligned}\int_\Omega \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\,\delta\boldsymbol{\nabla}\phi_\alpha\, d^4x &= \int_{t_1}^{t_2}\oint_{\partial\mathcal{V}} \delta\phi_\alpha\,\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\cdot d\mathbf{a}\, dt\\ &\quad - \int_\Omega \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\right)\delta\phi_\alpha\, d^4x\,. \end{aligned} \tag{18}$$

Once again, since we have fixed the boundary conditions, $\delta\phi_\alpha\big|_{\partial\mathcal{V}} = 0$. Thus,

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\,\delta\boldsymbol{\nabla}\phi_\alpha\, d^4x = -\int_\Omega \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\right)\delta\phi_\alpha\, d^4x\,. \tag{19}$$

By implementing Eqs. (15) and (19) into Eq. (13), we obtain

$$\int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \phi_\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\right)\right]\delta\phi_\alpha\, d^4x = 0. \tag{20}$$

As the equation must hold for every domain of integration, the integrand must be zero. As the equation also must hold for every possible combination of variations in the fields $\delta\phi_\alpha$ we finally conclude that

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\phi_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0. \tag{21}$$

As an example, let us consider the equations of motion described by the Lagrangian[2] for the vibrating string, obtained in Section I:

$$\mathcal{L} = \frac{\mu}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial\phi}{\partial x}\right)^2. \tag{22}$$

Thus, we have

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial\phi} = 0, \\ \dfrac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \mu\partial_t\phi, \\ \dfrac{\partial\mathcal{L}}{\partial(\partial_x\phi)} = -\tau\partial_x\phi, \end{cases} \tag{23}$$

where we denote a partial derivative with respect to a variable u as ∂_u. Substitution in the Euler-Lagrange Equations yields

$$\mu\frac{\partial^2\phi}{\partial t^2} - \tau\frac{\partial^2\phi}{\partial x^2} = 0, \tag{24}$$

which is, indeed, the wave equation satisfied by the string.

III. RELATIVISTIC FIELD THEORIES

One might be interested in consider field theories under the light of the Theory of Relativity. It would then be interesting for us to drop description in terms of time and, instead, consider the coordinate $x^0 = ct$, allowing us to treat space-time in a coherent way.

Notice that the Euler-Lagrange Equations are kept unchanged under the scale transformation $t \to x^0 = ct$, for

$$\frac{\partial}{\partial x^0}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi_\alpha/\partial x^0)}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(c\cdot\frac{\partial\mathcal{L}}{\partial(\partial\phi_\alpha/\partial t)}\right). \tag{25}$$

Thus, in terms of x^0, we may write the Euler-Lagrange Equations as

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_\alpha} = 0, \tag{26}$$

where, as usual,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{27}$$

[2] Sometimes we shall refer to the Lagrangian density as simply "Lagrangian". The difference should be made clear by the context.

For a given value of α, if ϕ_α is a scalar field it will hold that $\partial_\mu\phi_\alpha$ is a one-form. In a similar manner, if the Lagrangian $\mathcal{L}$ is a scalar, then $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}$ is a four-vector[3] and the first-term on the Euler-Lagrange Equations is a scalar. Therefore, if we want the Euler-Lagrange Equations to be covariant, we must simply demand $\mathcal{L}$ to be a scalar. As d^x, the four-dimensional volume element, is invariant under Lorentz Transformations, the action $S = \int\mathcal{L}\,d^x$ will be a scalar whenever the Lagrangian is a scalar[1, 2], and thus the requirement of S to be a scalar is another argument in defense of $\mathcal{L}$ being a scalar quantity.

A. Maxwell's Electrodynamics

As an example of a Relativistic Field Theory, we may consider the Lagrangian for Maxwell's Electrodynamics[4]:

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}J^\mu A_\mu, \tag{28}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{29}$$

The four-vectors J^μ and A^μ are, in fact, covariant forms of familiar quantities. Namely, they are the four-current and four-potential, composed from the charge density ρ, vector current density $\mathbf{J}$, scalar potential V and vector potential $\mathbf{A}$ as

$$\begin{cases} J^\mu = (c\rho, \mathbf{J}), \\ A^\mu = (V, \mathbf{A}). \end{cases} \tag{30}$$

We are now dealing with the fields $\phi_\alpha \equiv A_\alpha$, where α is a four-vector index. Our notation will be simplified if we write

$$A_{\mu\alpha} \equiv \partial_\mu A_\alpha. \tag{31}$$

As $\frac{\partial\mathcal{L}}{\partial A_\alpha} = \frac{1}{c}J^\alpha$, the Euler-Lagrange Equations become

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial A_{\mu\alpha}}\right) - \frac{1}{c}J^\alpha = 0. \tag{32}$$

[3] Some texts refer to four-vectors as *contravariant* (four-)vectors and to one-forms as *covariant* (four-)vectors. As I am unable to remember which one is which, and therefore I shall stick to calling them four-vectors and one-forms.

[4] As you might know from Classical Electrodynamics, the system of units we choose may change the way we express our equations. We are sticking to Gaussian units. Furthermore, as you might know from Relativity, the metric convention we choose may also change the way we express our equations. We are sticking to the $(-+++)$ metric signature.

We then make

$$\frac{\partial \mathcal{L}}{\partial A_{\mu\alpha}} = \frac{-1}{16\pi}\left(\frac{\partial F_{\tau\nu}}{\partial A_{\mu\alpha}}F^{\tau\nu} + F_{\tau\nu}\frac{\partial F^{\tau\nu}}{\partial A_{\mu\alpha}}\right). \tag{33}$$

But

$$\begin{aligned} F_{\tau\nu}\frac{\partial F^{\tau\nu}}{\partial A_{\mu\alpha}} &= g_{\tau\beta}g_{\nu\gamma}F^{\beta\gamma}\frac{\partial F^{\tau\nu}}{\partial A_{\mu\alpha}}, \\ &= g_{\beta\tau}g_{\gamma\nu}\frac{\partial F^{\tau\nu}}{\partial A_{\mu\alpha}}F^{\beta\gamma}, \\ &= \frac{\partial F_{\beta\gamma}}{\partial A_{\mu\alpha}}F^{\beta\gamma}, \end{aligned} \tag{34}$$

where $g_{\mu\nu}$ is the metric tensor. Therefore,

$$\frac{\partial \mathcal{L}}{\partial A_{\mu\alpha}} = \frac{-1}{8\pi}\frac{\partial F_{\tau\nu}}{\partial A_{\mu\alpha}}F^{\tau\nu}. \tag{35}$$

However, we already know that $F_{\mu\nu} = A_{\mu\nu} - A_{\nu\mu}$. Thus,

$$\begin{aligned} \frac{\partial F_{\tau\nu}}{\partial A_{\mu\alpha}} &= \delta^{\mu}{}_{\tau}\delta^{\alpha}{}_{\nu} - \delta^{\mu}{}_{\nu}\delta^{\alpha}{}_{\tau}, \\ \frac{\partial F_{\tau\nu}}{\partial A_{\mu\alpha}}F^{\tau\nu} &= \delta^{\mu}{}_{\tau}\delta^{\alpha}{}_{\nu}F^{\tau\nu} - \delta^{\mu}{}_{\nu}\delta^{\alpha}{}_{\tau}F^{\tau\nu}, \\ &= F^{\mu\alpha} - F^{\alpha\mu}, \\ &= 2F^{\mu\alpha}, \\ \frac{\partial \mathcal{L}}{\partial A_{\mu\alpha}} &= -\frac{1}{4\pi}F^{\mu\alpha}, \\ \frac{\partial \mathcal{L}}{\partial A_{\mu\alpha}} &= \frac{1}{4\pi}F^{\alpha\mu} \end{aligned} \tag{36}$$

Substitution in Eq. (32) yields

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}J^{\mu}, \tag{37}$$

which is the covariant form of the inhomogeneous Maxwell's Equations. Indeed, suppose $\mu = 0$. Then, since $J^0 = c\rho$ (Eq. (30)),

$$\begin{aligned} \partial_\nu F^{0\nu} &= 4\pi\rho, \\ \partial_\nu\left(\partial^0 A^\nu - \partial^\nu A^0\right) &= 4\pi\rho, \\ \partial_\nu\left(\frac{1}{c}\frac{\partial}{\partial t}A^\nu + \partial^\nu V\right) &= 4\pi\rho. \end{aligned} \tag{38}$$

Opening up the derivatives,

$$\begin{aligned} 4\pi\rho &= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial V}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A}\right) + \nabla^2 V - \frac{1}{c}\frac{\partial^2 V}{\partial t^2}, \\ &= -\boldsymbol{\nabla}\cdot\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}V\right). \end{aligned} \tag{39}$$

We know from Classical Electrodynamics[3, 4] that the electric field, $\mathbf{E}$, is given in terms of the potentials V and $\mathbf{A}$ by

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{40}$$

Therefore, we obtain

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho, \tag{41}$$

which is Gauss's Law.

If, on the other hand, we pick $\mu = i, i = 1, 2, 3$, it follows from the fact that $J^i = J_i$, where J_i denotes the i-th component of the current density vector $\mathbf{J}$, that

$$\begin{aligned} \frac{4\pi}{c}J_i &= \partial_\nu F^{i\nu}, \\ &= \partial_\nu\left(\partial^i A^\nu - \partial^\nu A^i\right), \\ &= \partial_\nu\left(\frac{\partial}{\partial x_i}A^\nu - \partial^\nu A_i\right), \\ &= \frac{\partial}{\partial x_i}\left(\frac{1}{c}\frac{\partial V}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A}\right) - \nabla^2 A_i + \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2}. \end{aligned} \tag{42}$$

If we join these components in a single vector-equation, we get

$$\begin{aligned} \frac{4\pi}{c}\mathbf{J} &= \boldsymbol{\nabla}\left(\frac{1}{c}\frac{\partial V}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A}\right) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}, \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}V\right), \\ &= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{43}$$

where we used that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times A) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$.

As we know from Classical Electrodynamics[3, 4] that the magnetic field $\mathbf{B}$ relates to the vector potential $\mathbf{A}$ through

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{44}$$

it follows then that

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}, \tag{45}$$

which is the Ampère-Maxwell Law.

It appears that the Euler-Lagrange Equations only led us to half of Maxwell's Equations, and therefore you are probably wondering where are the others! They come in fact from the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Notice that

$$\begin{aligned} \partial_\tau F_{\mu\nu} &= \partial_\tau\partial_\mu A_\nu - \partial_\tau\partial_\nu A_\mu, \\ &= \partial_\tau\partial_\mu A_\nu - \partial_\nu\partial_\tau A_\mu. \end{aligned} \tag{46}$$

Thus, it follows that

$$\partial_\tau F_{\mu\nu} + \partial_\mu F_{\nu\tau} + \partial_\nu F_{\tau\mu} = \partial_\tau \partial_\mu A_\nu - \partial_\nu \partial_\tau A_\mu + \partial_\mu \partial_\nu A_\tau - \partial_\tau \partial_\mu A_\nu + \partial_\nu \partial_\tau A_\mu - \partial_\mu \partial_\nu A_\tau. \quad (47)$$

As you can see, every term on the right-hand side of the expression cancels out and we are left with

$$\partial_\tau F_{\mu\nu} + \partial_\mu F_{\nu\tau} + \partial_\nu F_{\tau\mu} = 0, \quad (48)$$

which I claim to be a covariant expression of the homogeneous Maxwell Equations.

Let $\mu = 1, \nu = 2, \tau = 3$ and $i, j, k = 1, 2, 3$. Then, if we open up Eq. (48) in terms of the four-potential (just like in Eq. (47)), we get

$$\begin{aligned} 0 &= \partial_3 \partial_1 A_2 - \partial_2 \partial_3 A_1 + \partial_1 \partial_2 A_3 \\ &\quad - \partial_3 \partial_1 A_2 + \partial_2 \partial_3 A_1 - \partial_1 \partial_2 A_3, \\ &= \partial_1 (\partial_2 A_3 - \partial_3 A_2) \\ &\quad + \partial_2 (\partial_3 A_1 - \partial_1 A_3) \\ &\quad + \partial_3 (\partial_1 A_2 - \partial_2 A_1), \\ &= \partial_i \left(\epsilon^{ijk} \partial_j A_k\right), \end{aligned} \quad (49)$$

where ϵ^{ijk} denotes the Levi-Civita symbol (with $\epsilon^{123} = 1$).

In vector notation, we get

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0. \quad (50)$$

This might seem obvious, but if we recall that $\mathbf{B} = \nabla \times \mathbf{A}$, it follows that

$$\nabla \cdot \mathbf{B} = 0, \quad (51)$$

which is Gauss's Law for Magnetism.

Finally, let $\mu = i, \nu = j, \tau = 0$. Once again, we let $i, j = 1, 2, 3$. Eq. (48) now yields

$$\begin{aligned} 0 &= \partial_0 \partial_i A_j - \partial_j \partial_0 A_i + \partial_i \partial_j A_0 \\ &\quad - \partial_0 \partial_i A_j + \partial_j \partial_0 A_i - \partial_i \partial_j A_0. \end{aligned} \quad (52)$$

This equation holds $\forall\, i, j = 1, 2, 3$. Let now k also assume the values $1, 2, 3$. We might now multiply the equation as a whole by a Levi-Civita symbol and sum over all repeated indexes, *i.e.*,

$$\begin{aligned} 0 &= \epsilon^{ijk} \partial_0 \partial_i A_j - \epsilon^{ijk} \partial_j \partial_0 A_i + \epsilon^{ijk} \partial_i \partial_j A_0 \\ &\quad - \epsilon^{ijk} \partial_0 \partial_i A_j + \epsilon^{ijk} \partial_j \partial_0 A_i - \epsilon^{ijk} \partial_i \partial_j A_0. \end{aligned} \quad (53)$$

Notice that

$$\begin{aligned} \epsilon^{ijk} \partial_0 \partial_i A_j &= \epsilon^{jik} \partial_0 \partial_j A_i, \\ &= -\epsilon^{ijk} \partial_0 \partial_j A_i. \end{aligned} \quad (54)$$

Furthermore,

$$\begin{aligned} \epsilon^{ijk} \partial_i \partial_j A_0 &= -\epsilon^{jik} \partial_i \partial_j A_0, \\ &= -\epsilon^{ijk} \partial_j \partial_i A_0, \\ &= -\epsilon^{ijk} \partial_i \partial_j A_0. \end{aligned} \quad (55)$$

Thus,

$$\begin{aligned} 0 &= \epsilon^{ijk} \partial_0 \partial_i A_j + \epsilon^{ijk} \partial_i \partial_0 A_j + \epsilon^{ijk} \partial_i \partial_j A_0 \\ &\quad + \epsilon^{ijk} \partial_0 \partial_j A_i + \epsilon^{ijk} \partial_j \partial_0 A_i + \epsilon^{ijk} \partial_i \partial_j A_0. \end{aligned} \quad (56)$$

After dividing the expression by 2, we obtain

$$\begin{aligned} 0 &= \epsilon^{ijk} \partial_0 \partial_i A_j + \epsilon^{ijk} \partial_j \partial_0 A_i + \epsilon^{ijk} \partial_i \partial_j A_0, \\ &= \epsilon^{ijk} \partial_0 \partial_i A_j - \epsilon^{ijk} \partial_i \partial_0 A_j + \epsilon^{ijk} \partial_i \partial_j A_0, \\ &= \epsilon^{ijk} \partial_i \left(\partial_j A_0 + \partial_0 A_j\right) - \epsilon^{ijk} \partial_i \partial_0 A_j, \\ &= \epsilon^{ijk} \frac{\partial}{\partial x_i} \left(\frac{\partial V}{\partial x_j} + \frac{1}{c} \frac{\partial A_j}{\partial t} \right) - \epsilon^{ijk} \frac{\partial}{\partial x_i} \left(\frac{1}{c} \frac{\partial A_j}{\partial t} \right). \end{aligned} \quad (57)$$

In vector notation,

$$\mathbf{0} = \nabla \times \left(\nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial c} \right) - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}). \quad (58)$$

If we recall one last time that $\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times A$, it follows that

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad (59)$$

which is Faraday's Law.

IV. NOETHER'S THEOREM

We know from Hamiltonian Mechanics that,

$$\frac{dH}{dt} = -\frac{\partial \mathfrak{L}}{\partial t}, \quad (60)$$

where H denotes the Hamiltonian[5] of a certain physical system described by the Lagrangian $\mathfrak{L}$. Therefore, whenever the Lagrangian does not depend explicitly on time, the Hamiltonian is a constant of motion. Many times, it happens that $H = E$, where E stands for energy, and we say that energy is conserved.

However, the most remarkable thing is that the reason for H to be conserved is that $\mathfrak{L}$ does not depend explicitly on time. Similarly, the canonical momentum

[5] You might wonder whether Field Theory can be studied from the Hamiltonian point of view. The answer is yes, and more details can be found about that in references [1, 2, 4].

$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ associated with the generalized coordinate q is also conserved whenever $\frac{\partial \mathcal{L}}{\partial q}$. The description of how these conservation laws arise (in Classical Field Theory) from symmetries in the Lagrangian is the main goal of this section.

We shall consider an infinitesimal transformation such that

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \Delta x^\mu, \\ \phi_\alpha(x) \to \phi'_\alpha(x') = \phi_\alpha(x) + \Delta\phi_\alpha(x). \end{cases} \tag{61}$$

We also introduce the notation for the variation due only to the change in form of ϕ_α:

$$\delta\phi_\alpha(x) = \phi'_\alpha(x) - \phi_\alpha(x). \tag{62}$$

Notice that

$$\begin{aligned} \Delta\phi_\alpha(x) &= \phi'_\alpha(x') - \phi_\alpha(x') + \phi_\alpha(x') - \phi_\alpha(x), \\ &= \delta\phi_\alpha(x') + \partial_\mu\phi_\alpha(x)\Delta x^\mu. \end{aligned} \tag{63}$$

If we ignore the second-order terms in the infinitesimal variations, it follows that

$$\Delta\phi_\alpha(x) = \delta\phi_\alpha(x) + \phi_{\alpha;\mu}(x)\Delta x^\mu. \tag{64}$$

We also notice for further use that, since Δ involves evaluating the fields at different points in space-time, $\Delta\partial_\mu \neq \partial_\mu\Delta$. Inserting $\phi_{\alpha;\nu}$ in Eq. (64) yields

$$\Delta\phi_{\alpha;\nu}(x) = \delta\phi_{\alpha;\nu}(x) + \phi_{\alpha;\nu;\mu}(x)\Delta x^\mu. \tag{65}$$

Nevertheless, δ simply involves evaluating the field before and after the transformation at the very same point, and therefore it commutes with ∂_μ.

Finally, the Lagrangian density will transform according to

$$\mathcal{L}(\phi_\alpha(x), \phi_{\alpha;\mu}(x), x) \to \mathcal{L}'(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x'). \tag{66}$$

Assuming space-time is flat, we are going to prove the existence of conserved quantities if this transformation satisfies the following properties

Form Invariance:: the functional form of the transformed Lagrangian is equal to the original one, *i.e.*

$$\mathcal{L}'(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x') = \mathcal{L}(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x'). \tag{67}$$

Invariance of the Action:: the numerical value of the action integral is not changed by the transformation, *i.e.*,

$$\Delta S = S' - S = 0, \tag{68}$$

where

$$\begin{cases} S' = \int_{\Omega'} \mathcal{L}'(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x')\, d^4x', \\ S = \int_{\Omega} \mathcal{L}(\phi_\alpha(x), \phi_{\alpha;\mu}(x), x)\, d^4x. \end{cases} \tag{69}$$

We must now compute how the Lagrangian is affected by the transformation. We are going to make a little notation abuse right now and write $\mathcal{L}' \equiv \mathcal{L}'(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x')$ for simplicity. Taking our first demand into consideration, we see that

$$\begin{aligned} \mathcal{L}' &= \mathcal{L}(\phi'_\alpha(x'), \phi'_{\alpha;\mu}(x'), x'), \\ &= \mathcal{L}(\phi_\alpha(x) + \Delta\phi_\alpha(x), \phi_{\alpha;\mu}(x) + \Delta\phi_{\alpha,\mu}(x), x + \Delta x), \\ &= \mathcal{L}(\phi_\alpha(x), \phi_{\alpha;\mu}(x), x) + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\Delta\phi_\alpha \\ &\quad + \frac{\partial\mathcal{L}}{\partial\phi_{\alpha;\mu}}\Delta\phi_{\alpha;\mu} + \frac{\partial\mathcal{L}}{\partial x^\nu}\Delta x^\nu. \end{aligned} \tag{70}$$

If we now consider Eqs. (64) and (65), it follows that

$$\begin{aligned} \mathcal{L}' &= \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\frac{\partial\phi_\alpha}{\partial x^\nu}\Delta x^\nu \\ &\quad + \frac{\partial\mathcal{L}}{\partial\phi_{\alpha;\mu}}\delta\phi_{\alpha;\mu} + \frac{\partial\mathcal{L}}{\partial\phi_{\alpha;\mu}}\frac{\partial\phi_{\alpha;\mu}}{\partial x^\nu}\Delta x^\nu + \frac{\partial\mathcal{L}}{\partial x^\nu}\Delta x^\nu, \\ &= \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial\phi_{\alpha;\mu}}\delta\phi_{\alpha;\mu} + \frac{d\mathcal{L}}{dx^\nu}\Delta x^\nu, \\ &\equiv \mathcal{L} + \delta\mathcal{L} + \frac{d\mathcal{L}}{dx^\nu}\Delta x^\nu. \end{aligned} \tag{71}$$

In the previous calculation, we write

$$\frac{d\mathcal{L}}{dx^\nu} \equiv \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\frac{\partial\phi_\alpha}{\partial x^\nu} + \frac{\partial\mathcal{L}}{\partial\phi_{\alpha;\mu}}\frac{\partial\phi_{\alpha;\mu}}{\partial x^\nu} + \frac{\partial\mathcal{L}}{\partial x^\nu} \tag{72}$$

to denote the partial derivative of the Lagrangian with respect to x^ν *without keeping the fields constant*, *i.e.*, we also consider the contribution the fields give to the derivative, as opposed to $\partial\mathcal{L}/\partial x^\nu$, in which we consider only the explicit dependence.

We must now change coordinates on the integral for S' in order to express ΔS as an integral over x. We have

$$d^4x' = \frac{\partial\left(x'^0, x'^1, x'^2, x'^3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)}\, d^4x. \tag{73}$$

In order to calculate the Jacobian determinant of the transformation $x^\nu \to x^\nu + \Delta x^\nu$, we must use the fact that given a matrix $J = \mathbb{1} + \epsilon A$, then

$$\det J = 1 + \epsilon \operatorname{tr} A + \mathcal{O}\left(\epsilon^2\right), \tag{74}$$

which is proved in the Appendix at page 11.

The Jacobian matrix for $x^\nu \to x^\nu + \Delta x^\nu$ can be written as $J = \mathbb{1} + J'$, where J' is the Jacobian matrix for the transformation $x^\nu \to \Delta x^\nu$. As Δx^ν is small, we can neglect higher-order terms when computing tha Jacobian determinant for $x^\nu \to x^\nu + \Delta x^\nu$. Thus, we may apply the aforementioned theorem in order to obtain

$$d^4x' = \left(1 + \frac{\partial \Delta x^\nu}{\partial x^\nu}\right) d^4x \,. \tag{75}$$

In view of Eqs. (71) and (75), the expression for the Invariance of the Action becomes

$$\int_\Omega \left(\mathcal{L} + \delta\mathcal{L} + \frac{d\mathcal{L}}{dx^\nu}\Delta x^\nu\right)\left(1 + \frac{\partial \Delta x^\nu}{\partial x^\nu}\right) - \mathcal{L}\, d^4x = 0. \tag{76}$$

If we ignore variations of second order and above, the expression simplifies to

$$\begin{aligned} \int_\Omega \mathcal{L}\frac{\partial \Delta x^\nu}{\partial x^\nu} + \delta\mathcal{L} + \frac{d\mathcal{L}}{dx^\nu}\Delta x^\nu\, d^4x = 0, \\ \int_\Omega \delta\mathcal{L} + \frac{d}{dx^\nu}(\mathcal{L}\Delta x^\nu)\, d^4x = 0. \end{aligned} \tag{77}$$

Notice that, by applying the Euler-Lagrange Equations[6]

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\,\delta\phi_\alpha + \frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\,\delta\phi_{\alpha;\mu}\,, \\ &= \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\nu}}\right)\delta\phi_\alpha + \frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\,\delta\phi_{\alpha;\mu}\,, \\ &= \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\nu}}\,\delta\phi_\alpha\right). \end{aligned} \tag{78}$$

If we substitute this result in Eq. (77), it follows that

$$\int_\Omega \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\,\delta\phi_\alpha + \mathcal{L}\Delta x^\nu\right) d^4x = 0. \tag{79}$$

Eq. (79) can be put in a more convenient form if we express $\delta\phi_\alpha$ and Δx^ν in terms of the infinitesimal parameters that characterize the transformation. Let us assume there are R such parameters, each one denoted by ϵ_r with $1 \leqslant r \leqslant R$. Then we can write

$$\begin{cases} \Delta x^\nu = X^{\nu(r)}\epsilon_r, \\ \Delta\phi_\alpha = \Psi_\alpha{}^{(r)}\epsilon_r. \end{cases} \tag{80}$$

One should be aware that, although the notation may suggest the opposite, the indices α and r are not necessarily tensor indices, but summation over repeated indices is implicit.

Due to Eq. (64), we get

$$\delta\phi_\alpha = \left(\Psi_\alpha{}^{(r)} - \phi_{\alpha;\nu}X^{\nu(r)}\right)\epsilon_r. \tag{81}$$

If we define

$$\Theta^{\mu(r)} = -\frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\left(\Psi_\alpha{}^{(r)} - \phi_{\alpha;\nu}X^{\nu(r)}\right) - \mathcal{L}X^{\mu(r)}, \tag{82}$$

then inserting Eqs. (80) and (81) into Eq. (79) yields

$$-\int_\Omega \frac{d\Theta^{\mu(r)}}{dx^\mu}\epsilon_r\, d^4x = 0, \tag{83}$$

which is valid for every possible combination of ϵ_r and any integration volume Ω. Due to this freedom, the integral can only vanish if the integrand vanishes as well, and therefore we obtain the following R continuity equations:[7]

$$\partial_\mu\Theta^{\mu(r)} = 0. \tag{84}$$

It is worth mentioning that these conserved currents also may imply the existence of conserved charges. Let $\Theta^{\mu(r)} = \left(\Theta^{0(r)}, \boldsymbol{\Theta}^{(r)}\right)$. Then we may write Eq. (84) in vector notation as

$$\frac{\partial\Theta^{0(r)}}{\partial x^0} + \boldsymbol{\nabla}\cdot\boldsymbol{\Theta}^{(r)} = 0. \tag{85}$$

Integration over a volume $\mathcal{V}$ yields

$$\begin{aligned} \frac{d}{dx^0}\int_\mathcal{V}\Theta^{0(r)}\, d^3x &= -\int_\mathcal{V}\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}^{(r)}\, d^3x\,, \\ &= -\oint_{\partial\mathcal{V}}\boldsymbol{\Theta}^{(r)}\cdot d\mathbf{a}\,. \end{aligned} \tag{86}$$

If $\mathcal{V}$ is the entire three-dimensional space and the fields vanish sufficiently fast as the coordinates grow towards infinity, the surface integral vanishes, implying the global conservation of the quantities

$$C^{(r)} = \int\Theta^{0(r)}\, d^3x\,, \tag{87}$$

which are known as Noether charges.

[6] You might be wondering why we are using d/dx^μ instead of $\partial/\partial x^\mu$ on the Euler-Lagrange Equations. If you take a look on our derivation of the equations, you will see that the derivatives we are considering there, albeit partial, do consider the indirect contributions due to the dependence of the fields on x^μ.

[7] Since we are not differentiating anything while holding the fields constant, it is possible to use the notation ∂_μ again without any risk of confusion.

V. APPLICATIONS OF NOETHER'S THEOREM

A. The Stress-Energy Tensor

As a first example, let us consider symmetry under a space-time translation, *i.e.*,

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}, \tag{88}$$

where ϵ^{μ} is a constant four-vector. The fields remain unchanged.

As ϵ^{μ} is a constant, the Jacobian of the transformation is unity. Since the fields are unchanged, the action will be kept invariant under this transformation if the Lagrangian does not depend explicitly on the coordinates.

Since

$$\begin{cases} \epsilon^{\mu} = X^{\mu(r)}\epsilon_r, \\ 0 = \Psi_{\alpha}{}^{(r)}\epsilon_r, \end{cases} \tag{89}$$

it follows that $X^{\mu\nu} = g^{\mu\nu}$ and $\Psi_{\alpha}{}^{(r)} = 0$. In this case, r does have tensor character. If we substitute these expressions into Eq. (82), we get

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\phi_{\alpha;\tau}g^{\tau\nu} - \mathcal{L}g^{\mu\nu}, \\ &= \frac{\partial \mathcal{L}}{\partial \phi_{\alpha;\mu}}\frac{\partial \phi_{\alpha}}{\partial x_{\nu}} - \mathcal{L}g^{\mu\nu}. \end{aligned} \tag{90}$$

The tensor $T^{\mu\nu}$ is called the *stress-energy tensor* or *energy-momentum tensor*.

The Noether charge associated to it is the four-vector given by

$$p^{\nu} = \int T^{0\nu}\, d^3x\,. \tag{91}$$

The zeroth component of $T^{0\nu}$ is given by

$$\begin{aligned} T^{00} &= -\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{\alpha}}\dot{\phi} + \mathcal{L}, \\ &= -\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{\alpha}}\dot{\phi} - \mathcal{L}\right). \end{aligned} \tag{92}$$

This is in fact minus the Hamiltonian density, which is used in the Hamiltonian treatment of Classical Field Theory[1, 2]. It is then natural to say that T^{00} is minus the energy density of the fields[5].

Therefore, p^0 is, up to a signal, the integral of the energy density of the fields, and therefore represent the energy stored in the fields. Due to covariance considerations and the knowledge that momentum is preserved under spatial translations, we conclude that $p^i, i = 1, 2, 3$ are the components of the fields' momentum. This establishes p^{μ} as the fields' four-momentum and justifies the term *energy-momentum* tensor.

B. The Conservation of Electric Charge

In order to obtain the conservation of electric charge, let us consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{93}$$

This Lagrangian describes the theory of Quantum Electrodynamics (QED)[6–8]. The term

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{94}$$

is the same we have seen in the Lagrangian for Maxwell's Electrodynamics. The missing 4π is due to a new choice of units: when dealing with High Energy Physics it is customary to adopt a system of units such that $\hbar = c = 1$. The term

$$\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \tag{95}$$

gives rise to the Dirac equation for the motion of a relativistic electron. Finally, the term

$$-e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \tag{96}$$

couples electrons and photons together, allowing them to interact with each other, and matches the $J^{\mu}A_{\mu}$ we had earlier, the sole difference being that we now are dealing with a specific current given by $J^{\mu} = -e\bar{\psi}\gamma^{\mu}\psi$.

γ^{μ} are four 4×4 matrices satisfying $\{\gamma^{\mu}, \gamma^{\nu}\} = -2g^{\mu\nu}$. A possible representation of these matrices is the Weyl, or chiral, representation, given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{97}$$

where σ_i are the Pauli matrices.

Since γ^{μ} are 4×4 matrices, it is not that surprising that ψ is a 4-component spinor. We define $\bar{\psi} = \psi^{\dagger}\gamma^0$.

Let now $\alpha(x)$ be some real function depending on the space-time coordinates. Consider the following transformation

$$\begin{cases} A'_{\mu} = A_{\mu} + \partial_{\mu}\alpha(x), \\ \psi' = e^{-ie\alpha(x)}\psi. \end{cases} \tag{98}$$

Symmetries that are parametrized by a function such as $\alpha(x)$ are said to be *gauge* of *local symmetries*. On the other hand, if we had a constant α instead, we would call it a *global symmetry*. Notice that every gauge symmetry implies a global symmetry.

Notice that $F_{\mu\nu}$ is left unchanged by this transformation:

$$\begin{aligned} F'_{\mu\nu} &= \partial_{\mu}A'_{\nu} - \partial_{\nu}A'_{\mu}, \\ &= \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\alpha(x) - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\alpha(x), \\ &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \end{aligned} \tag{99}$$

Therefore, the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ of the QED Lagrangian is left unchanged by the transformation.

In fact, the Lagrangian as a whole is left unchanged as well. Indeed,

$$\begin{aligned}
\mathcal{L}' &= -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \bar{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' - e\bar{\psi}'\gamma^\mu\psi' A'_\mu, \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}e^{+ie\alpha(x)}(i\gamma^\mu\partial_\mu - m)e^{-ie\alpha(x)}\psi \\
&\quad - e\bar{\psi}e^{+ie\alpha(x)}\gamma^\mu e^{-ie\alpha(x)}\psi A_\mu \\
&\quad - e\bar{\psi}e^{+ie\alpha(x)}\gamma^\mu e^{-ie\alpha(x)}\psi\partial_\mu\alpha(x), \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha(x) \\
&\quad - e\bar{\psi}\gamma^\mu\psi A_\mu - e\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha(x), \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu, \\
&= \mathcal{L}.
\end{aligned} \tag{100}$$

Thus, we have found a gauge symmetry in our Lagrangian. This also implies a global symmetry by choosing $\alpha(x) = \alpha \in \mathbb{R}$. Since ψ is multiplied by $e^{-ie\alpha}$, with $\alpha \in \mathbb{R}$, the symmetry group associated to this gauge transformation is $U(1)$[9].

If we pick an infinitesimal α, we get the following transformation for the fields

$$\begin{cases} A'_\mu = A_\mu, \\ \psi' = (1 - ie\alpha)\,\psi, \end{cases} \tag{101}$$

with the space-time coordinates left unchanged. This yields

$$\begin{cases} X^{\nu(r)} = 0, \\ \Psi_a = -ie\psi_a, \end{cases} \tag{102}$$

where a stands for the spinorial index of ψ.

The Noether current is then given by[8]

$$\begin{aligned}
\Theta^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Psi, \\
&= ie\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi, \\
&= -e\bar{\psi}\gamma^\mu\psi.
\end{aligned} \tag{103}$$

However, this is simply the electric current density we chose at the beginning of this section, *i.e.*,

$$J^\mu = -e\bar{\psi}\gamma^\mu\psi. \tag{104}$$

Thus, J^μ is the Noether current associated with $U(1)$ gauge symmetry, and we may conclude that

$$\partial_\mu J^\mu = 0. \tag{105}$$

One should notice that this derivation was made possible when we coupled the electromagnetic field, described by A_μ, to a matter field ψ responsible for carrying the electric charge. Furthermore, we could also couple some more fields (corresponding, *e.g.*, to muons and taus) and obtain a more robust result. The current expression we have yields the conservation law for charge carried by electrons and positrons only and ignores any other charged particles.

It is also worth mentioning that $\bar{\psi}\gamma^\mu\psi$ is the probability four-current density for the Dirac Equation[10]. Therefore, the electric four-current density is nothing but the electron's charge multiplied by its probability four-current density.

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[8] One might do the calculation considering each spinorial component of ψ and then see that formally differentiating with respect to $\partial_\mu\psi$ yields the same result.

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Appendix: Determinant of a Near-Identical Matrix

Theorem 1:
Let $J = \mathbb{1} + \epsilon A$ *be a* $N \times N$ *matrix. Then it holds that*

$$\det J = 1 + \epsilon \operatorname{tr} A + \mathcal{O}\left(\epsilon^2\right). \qquad \square$$

Proof: Let us write e_i for the elements of the canonical basis, *i.e.*, e_i is the i-th column of $\mathbb{1}$. In a similar manner, let us denote the i-th column of A by A_i.

The determinant is a multilinear form on the columns of J. Therefore,

$$\begin{aligned}
\det J &= \det\left(e_1 + \epsilon A_1, e_2 + \epsilon A_2, \ldots, e_N + \epsilon A_N\right), \\
&= \det\left(e_1, e_2, \ldots, e_N\right) + \epsilon \det\left(A_1, e_2, \ldots, e_N\right) \\
&\quad + \epsilon \det\left(e_1, A_2, \ldots, e_N\right) + \cdots \\
&\quad + \epsilon \det\left(e_1, e_2, \ldots, A_N\right) + \mathcal{O}\left(\epsilon^2\right), \\
&= 1 + \epsilon\left(A_{11} + A_{22} + \cdots + A_{NN}\right) + \mathcal{O}\left(\epsilon^2\right), \\
&= 1 + \epsilon \operatorname{tr} A + \mathcal{O}\left(\epsilon^2\right). \qquad \blacksquare
\end{aligned}$$